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ANALYSIS OF THE FLUCTUATION OF RESONANCE DRIVING TERMS FOR NONLINEAR LATTICE OPTIMIZATION

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Abstract

Minimizing resonance driving terms (RDTs) of nonlinear magnets is a traditional approach to enlarge the dynamic aperture (DA) of a storage ring. The local cancellation of nonlinear dynamics, which is adopted by some diffraction-limited storage rings, is more effective than the global cancellation. The former has smaller fluctuation of RDTs along the ring. In this paper, the correlation between two kinds of RDT fluctuations is found. The physical analysis shows that minimizing the RDT fluctuations is beneficial for controlling the crossing terms and thus enlarging the DA. This physical analysis is supported by the statistical analysis of nonlinear solutions of a double-bend achromat lattice.

INTRODUCTION

The widely-used analytical approach for the nonlinear optimization of storage rings is to minimize resonance driving terms (RDTs) of nonlinear magnets. In this approach, the Hamiltonian for particle motion is split into linear and nonlinear parts, and the nonlinear parts is expanded as the resonance basis, i.e. RDTs [1]. Minimizing the RDTs can control the corresponding resonance and thus enlarge the dynamic aperture (DA). The local nonlinear cancellation, which is used in the lattice design of some diffraction-limited storage rings, is more effective than the global cancellation [2]. And the former has smaller fluctuation of RDTs along the ring. There are two ways to calculate the longitudinal fluctuation of RDTs. One is to calculate the accumulated RDTs with a fixed starting position, and the build-up and cancellation of RDTs are shown in this way. We call it the RDT build-up fluctuation. We have shown that minimizing the RDT build-up fluctuations is more effective than minimizing the commonly used one-turn RDTs in enlarging the DA [3]. The other is to calculate the one-turn map (or one-period map) with varying longitudinal starting position [4]. In this paper, we will study the correlation between these two kinds of RDT fluctuations. And then we will analyze the effects of minimizing the RDT fluctuations.

RELATION BETWEEN TWO KINDS OF RDT FLUCTUATIONS

For a storage ring lattice with N sextupoles, the one-period map observed at s_0 is

$$\mathcal{M}(s_0) = \mathcal{A}_{s_0}^{-1} e^{:h:} \mathcal{R} \mathcal{A}_{s_0}, \tag{1}$$

where $\mathcal{A}_{s_0}$ is a normalizing map, $\mathcal{R}$ is a rotation, and $e^{:h:}$ is the nonlinear Lie map. For the on-momentum particles, the n-th order generator of $e^{:h:}$ can be expanded as:

$$h_n = \sum_{j+k+l+m=n} h_{jklm} h_x^{+j} h_x^{-k} h_y^{+l} h_y^{-m}, \tag{2}$$

where $h_x^{\pm} \equiv \sqrt{2J_x} e^{\pm i\phi_x}$, $h_y^{\pm} \equiv \sqrt{2J_y} e^{\pm i\phi_y}$, with (J, ϕ) being action-angle variables, and h_{jklm} is the driving terms. For any thin sextupole a, its normalized Hamiltonian $\hat{V}_a$ can be expanded in the same way:

$$\hat{V}_a = \sum_{j+k+l+m=3} h_{a,jklm} h_x^{+j} h_x^{-k} h_y^{+l} h_y^{-m}. \tag{3}$$

For the third-order RDTs of one-period map, we have $h_{jklm} = \sum_{a=1}^{N} h_{a,jklm}$. The build-up fluctuation $h_{1\to t,jklm} \equiv \sum_{a=1}^{t} h_{a,jklm}$ shows the accumulated RDTs from s_0 to the t-th sextupole.

For the case of multiple periods, the accumulated RDTs from s_0 to t-th sextupole in the $(u+1)$-th period is:

$$\begin{aligned}
\sum_{a=1}^{uN+t} h_{a,jklm} &= \sum_{a=1}^{uN} h_{a,jklm} + \sum_{a=uN+1}^{uN+t} h_{a,jklm} \\
&= \sum_{a=1}^{N} h_{a,jklm} \frac{1 - e^{iu\boldsymbol{m}\cdot\boldsymbol{\mu}}}{1 - e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}}} + \sum_{a=1}^{t} h_{a,jklm} e^{iu\boldsymbol{m}\cdot\boldsymbol{\mu}} \\
&= \frac{\sum_{a=1}^{N} h_{a,jklm}}{1 - e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}}} - \left(\frac{\sum_{a=1}^{N} h_{a,jklm}}{1 - e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}}} - \sum_{a=1}^{t} h_{a,jklm} \right) e^{iu\boldsymbol{m}\cdot\boldsymbol{\mu}},
\end{aligned} \tag{4}$$

where $\boldsymbol{m} = (j-k, l-m)$ is the mode of resonance and $\boldsymbol{\mu} = (\mu_x, \mu_y)$ is the phase advances of one period. The third-order RDT build-up fluctuation can be written in the form of $C_{0,\boldsymbol{m}} + C_{t,\boldsymbol{m}} e^{iu\boldsymbol{m}\cdot\boldsymbol{\mu}}$, which is a circle in the complex plane when u is a variable. And $C_{t,\boldsymbol{m}}$ is dependent on the sextupole index t, so the build-up fluctuation of the RDT h_{jklm} is a series of concentric circles in the complex plane.

The second kind of RDT fluctuations shows the period map observed at different longitudinal positions. And we can measure it on a real machine. In order to measure the RDTs, we need another transformation to find the nonlinear invariants [4]:

$$e^{:-F:} e^{:h:} \mathcal{R} e^{:F:} = e^{:\mathrm{H}:} \mathcal{R}, \tag{5}$$

where H is the phase-independent Hamiltonian in normal forms, and F is such a transformation. When the observation position s is between n-th and $(n+1)$-th sextupole, the third

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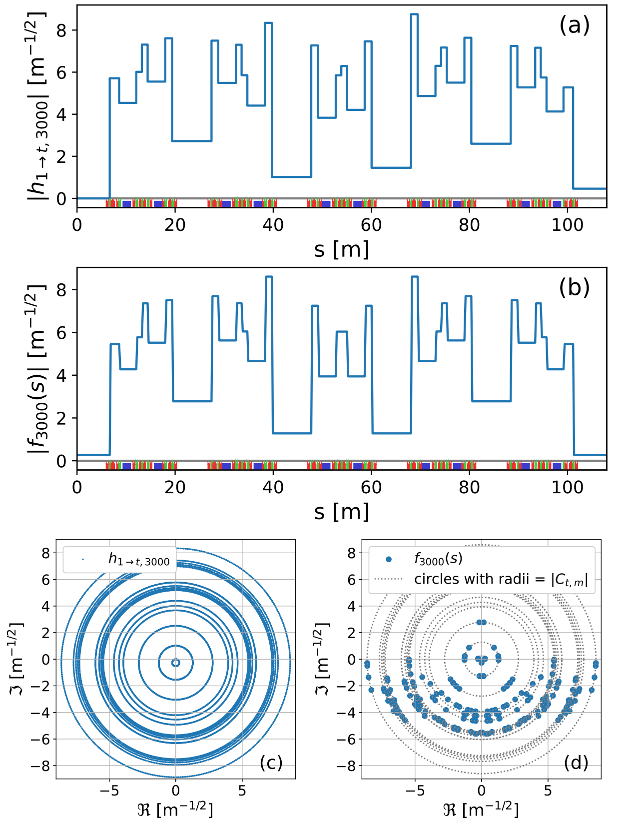

Figure 1: Correlation between two kinds of RDT fluctuations of the SSRF storage ring lattice. (a) The build-up fluctuation of h_{3000} in one SP. (b) The fluctuation of $f_{3000}(s)$ in one SP. (c) The build-up fluctuation of h_{3000} for 100 turns in the complex plane. We calculate multiple turns in order to make concentric circles visible. (d) The RDT $f_{3000}(s)$ in the complex plane.

order terms in F are:

$$\begin{aligned} f_{jklm}(s) &= \frac{\sum_{a=n+1}^{n+N} h_{a,jklm} e^{-i\boldsymbol{m}\cdot\Delta\boldsymbol{\phi}}}{1-e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}}} \\ &= \frac{\left(\sum_{a=1}^{N}-\sum_{a=1}^{n}+\sum_{a=N+1}^{N+n}\right) h_{a,jklm} e^{-i\boldsymbol{m}\cdot\Delta\boldsymbol{\phi}}}{1-e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}}} \\ &= \frac{\sum_{a=1}^{N} h_{a,jklm}-(1-e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}})\sum_{a=1}^{n} h_{a,jklm}}{1-e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}}} e^{-i\boldsymbol{m}\cdot\Delta\boldsymbol{\phi}}, \end{aligned} \tag{6}$$

where $\Delta\boldsymbol{\phi}$ is the phase advances between the observation position s and s_0. And we can see that $|f_{jklm}(s)|$ equals to $|C_{t,\boldsymbol{m}}|$. So these two kinds of RDT fluctuations are strongly related.

Figure 1(a) and 1(b) show these two kinds of RDT fluctuations for one super-period (SP) of the SSRF storage ring lattice. The SSRF storage ring consists of 4 SPs, each with 5 double-bend achromat (DBA) cells [5]. The term h_{3000} is almost cancelled after one SP. The term $C_{0,\boldsymbol{m}}$ in Eq. (4) is small and the two kinds of RDT fluctuations are almost the same. Figure 1(c) shows the build-up fluctuation in the complex plane. We calculate multiple turns in order to make the concentric circles visible. These circles have the same center $C_{0,\boldsymbol{m}}$ and different radii $|C_{t,\boldsymbol{m}}|$. In Fig. 1(d), $f_{jklm}(s)$ is also plotted in the complex plane. And we can see that they are on the circles with radii = $|C_{t,\boldsymbol{m}}|$. The fourth-order cases are more complex, but the relation still exists. For a fourth-order resonance $\boldsymbol{m}=\boldsymbol{m}_1+\boldsymbol{m}_2$, where $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are third-order resonances, the RDT build-up fluctuation has the form of $C_{0,\boldsymbol{m}}+C_{t,\boldsymbol{m}}e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}}+C'_{t,\boldsymbol{m}_1}e^{i\boldsymbol{m}_1\cdot\boldsymbol{\mu}}+C'_{t,\boldsymbol{m}_2}e^{i\boldsymbol{m}_2\cdot\boldsymbol{\mu}}$, and the fourth-order $|f_{jklm}(s)|$ still equals to $|C_{t,\boldsymbol{m}}|$.

EFFECTS OF MINIMIZING RDT FLUCTUATIONS

Physical Analysis

Since the RDTs are derived from the one-period map, it seems the build-up and cancellation of RDTs in the period is not important. For an ideal cancellation situation, e.g. two identical thin sextupoles separated by $-\mathcal{I}$ transformation, the nonlinear effects are completely cancelled. And the fluctuation of RDTs, which is determined by the strength of sextupoles, does not affect the cancellation. However, when the thickness of sextupoles is considered, an "error map" containing higher-order nonlinear terms appears. Reducing the strength of sextupoles can control the error map as well as the RDT fluctuations. The calculation of these two examples can be found in Ref. [6] (pages 146-149).

We think the cross-talk effect of sextupoles is the key. The perturbations of sextupoles drives fourth-order resonances by the cross-talk effect:

$$h_4=\sum_{b>a=1}^{N}\left[\hat{V}_a,\hat{V}_b\right]=\sum_{b=2}^{N}\left[\sum_{a=1}^{b-1}\hat{V}_a,\hat{V}_b\right], \tag{7}$$

and $\sum_{b>a=1}^{t}\left[\hat{V}_a,\hat{V}_b\right]$ shows the build-up fluctuations of fourth-order RDTs. The third-order RDT build-up fluctuations are involved in the calculation of fourth-order RDTs. Minimizing RDT fluctuations is beneficial for controlling the crossing terms and thus controlling the fourth-order resonances. Moreover, the cross-talk effect can also generate the fifth- and higher-order RDTs. So minimizing the RDT fluctuations is beneficial for controlling higher-order resonances. And this is the physics why minimizing the RDT fluctuations is more effective than minimizing the one-turn RDTs in enlarging the DA.

Statistical Analysis

We will then use numerous nonlinear lattice solutions to demonstrate this physical analysis. Some indicators need to be defined. The RDT h_{jklm} of the one-turn map is denoted as $h_{jklm,\text{ring}}$. And when all the n-th order one-turn RDTs are considered together, we have $h_{n,\text{ring}}=\sqrt{\sum_{j+k+l+m=n}h^2_{jklm,\text{ring}}}$. Since these two kinds of RDT fluctuations are related by $|C_{t,\boldsymbol{m}}|$, we can use $h_{jklm,\text{ave}}=\sum_{t=1}^{N}|C_{t,\boldsymbol{m}}|/N$ to show the RDT fluctuations. And similarly the n-th order RDT fluctuations is denoted as $h_{n,\text{ave}}=$

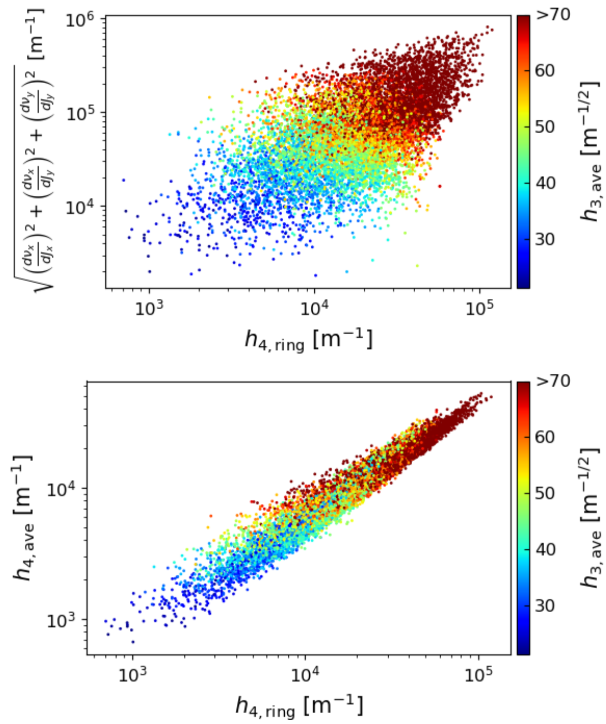

Figure 2: Correlation between the third-order RDT fluctuations $h_{3,\mathrm{ave}}$ and the crossing terms. The ADTS terms, the fourth-order one-turn RDTs $h_{4,\mathrm{ring}}$ and the fourth-order RDT fluctuations $h_{4,\mathrm{ave}}$ are generated by the cross-talk effect of sextupoles.

$\sqrt{\sum_{j+k+l+m=n} h^2_{jklm,\mathrm{ave}}}$. The correlation between these indicators can be measured using the Spearman rank-order correlation coefficient, which is a nonparametric measure of the monotonicity of the relationship between two datasets [7]. Like other correlation coefficients, it varies between -1 and +1 with 0 implying no correlation. Correlations of -1 or +1 imply an exact monotonic relationship.

The SSRF lattice is also taken as the example. The strengths of sextupoles are changed to produce different nonlinear solutions, with the horizontal and vertical chromaticities corrected to $(1, 1)$. A set of nonlinear solutions were generated randomly. Figure 2 shows the correlation between the third-order RDT fluctuations and the crossing terms. Both the fourth-order RDTs and the amplitude-dependent tune shift (ADTS) terms are generated by the sextupole crossing terms. We see that as the third-order RDT fluctuations reduce, the ADTS terms, the fourth-order one-turn RDTs and the fourth-order RDT fluctuations also roughly reduce. The Spearman correlation coefficient between $h_{3,\mathrm{ave}}$ and $h_{4,\mathrm{ring}}$ is 0.75, and it is 0.76 for $h_{3,\mathrm{ave}}$ and $h_{4,\mathrm{ave}}$, and 0.82 for $h_{3,\mathrm{ave}}$ and ADTS terms. All indicate strong correlations. Therefore, minimizing the RDT fluctuations can effectively control the crossing terms.

Figure 3 shows the correlation between the third-order one-turn RDTs $h_{3,\mathrm{ring}}$, the third-order RDT fluctuations $h_{3,\mathrm{ave}}$ and the DA area for a set of optimized nonlinear solutions. We see that the colors, which represent the DA areas, are roughly horizontally layered, and the red dots with large DAs sink to the bottom. This distribution indicates that minimizing $h_{3,\mathrm{ave}}$ is more effective than minimizing $h_{3,\mathrm{ring}}$ in enlarging the DA. The Spearman correlation coefficient between $h_{3,\mathrm{ave}}$ and DA area is -0.87, indicating a very strong correlation.

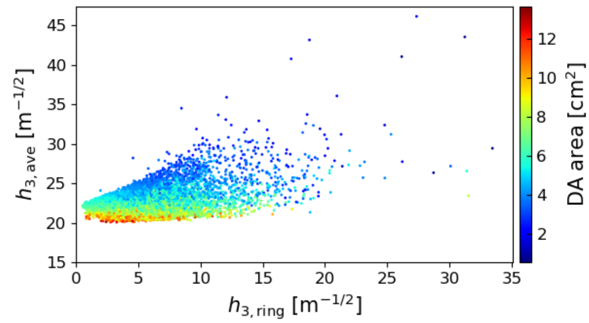

Figure 3: Correlation between the third-order RDT fluctuations $h_{3,\mathrm{ave}}$, the third-order one-turn RDTs $h_{3,\mathrm{ring}}$ and the DA area.

CONCLUSION

The local cancellation of nonlinear effects is more effective than the global cancellation. The former has smaller longitudinal RDT fluctuations, which means minimizing RDT fluctuation can be beneficial for enlarging the DA. The relation between two kinds of longitudinal RDT fluctuations was found in this paper. The physical analysis showed that minimizing the RDT fluctuations is beneficial for controlling the crossing terms, which drive the higher-order resonances and ADTS. Therefore, minimizing RDT fluctuations is beneficial for optimizing the nonlinear dynamics and enlarging the DA. This physical analysis was demonstrated by the statistical analysis of nonlinear solutions of a DBA lattice. The code for calculating the RDT fluctuations was shared on a github page [8].

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