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LOWEST LONGITUDINAL AND TRANSVERSE RESISTIVE-WALL WAKE AND IMPEDANCE FOR NONULTRA-RELATIVISTIC BEAMS*

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Abstract

With the development of the steady state micro bunching (SSMB) storage ring, its parameters reveal that the ultra-relativistic assumption which is wildly used is not valid for the electron beam bunch train, which has length in the 100nm range, spacing of 1um and energy in hundreds MeV range. The strength of the interaction between such bunches and the potential instability may need careful evaluation. At the same time, the effect of the space charge inside a single bunch due to space charge effect also needs to be considered. In this article, we reorganized the lowest-order longitudinal wakefield under non-ultra relativistic conditions, and modified the inconsistent part in the theoretical derivation in some essays of the lowest-order transverse wakefield. We present the modified theoretical results and analysis. The action area are then divided into three parts. It lays foundation in future research.

INTRODUCTION

Based on published literature, we have reorganized the lowest-order longitudinal wakefield under nonultra-relativistic conditions which is the monopole longitudinal wakefield, and we have modified the inconsistent part in the theoretical derivation of the lowest-order transverse wakefield which is the dipole transverse wakefield in existing literature. We present the modified theoretical results and analysis. The calculation results are evaluated, and the action area of the non-ultra relativistic wakefield is divided into a short-range dominated by the source charge space force, a middle section dominated by the mirror space charge force, and a long-range resistive wall that can be estimated using classical ultra-relativistic assumption. This lays the foundation and clarifies the ideas for subsequent beam dynamics analysis.

LONGITUDINAL WAKEFIELD

Previous Result

The classical Longitudinal Wakefield for multipole has an analytical expression [1]

$$W_m(z) = -\frac{c}{\pi b^{2m+1}(1+\delta_{m0})}\sqrt{\frac{Z_0}{\pi\sigma_c}}\frac{L}{|z|^{\frac{1}{2}}} \tag{1}$$

$$W_m'(z) = -\frac{c}{2\pi b^{2m+1}(1+\delta_{m0})}\sqrt{\frac{Z_0}{\pi\sigma_c}}\frac{L}{|z|^{\frac{3}{2}}} \tag{2}$$

where b is the radius of the pipe, σ_c is the conductivity of the surrounding medal. This result is obtained by ultra-relativistic limit $\gamma \to +\infty$,e.g. the speed of the electron is the speed of light c. Meanwhile, the effective region of the longitudinal coordinate is

$$\chi^{\frac{1}{3}}b \ll |z| \ll \chi^{-\frac{1}{3}}\frac{c}{b},\ \ z<0 \tag{3}$$

where $\chi = \frac{1}{\mu\sigma_c bc}$. For Aluminum pipe whose radius is in several centimeters range, the effective region will be $1.95\times10^{-5}m \ll z \ll 1.54\times10^{13}m$.

In order to calculate short range wakefield, in SLAC-PUV-95-7074 [2] there is a formula for longitudinal Electric Field in the time-space domain

$$E_z^m(z) = -16\gamma \left(\frac{1}{3}e^{-\gamma^{\frac{2}{3}}\frac{z}{s_0}}\cos\frac{\sqrt{3}\gamma^{\frac{2}{3}}z}{s_0} - \frac{\sqrt{2}\gamma}{\pi}\int_0^\infty \frac{x^2e^{-x^2\frac{z}{s_0}}}{x^6+8\gamma^2}\right) \tag{4}$$

where

$$s_0 = b^{\frac{1}{2}}\left(\frac{c}{2\pi\sigma}\right)^{\frac{1}{3}} \tag{5}$$

this equation is valid for all $z \le 0$.

SSMB Case Monopole Wake Benchmark

For the SSMB Parameters showed at Table 1 reveal that the space charge effect estimate by $\frac{2b}{\gamma}$ would be about 30.6 µm. Such an effective length would be much larger than the spacing between bunch to bunch. So, it would be better that we consider if it is appropriate to view space charge electromagnet field as a round plate.

Table 1: SSMB Bunch Train Parameters

Parameter	Value	Purpose
Length	10 nm	Longitudinal Coherent
Transverse size	10–100 µm	
Spacing	1 µm	High Average Power
Energy	250 MeV	$\gamma \approx 490.2$

There is such a monopole result derived for nonultra-relativistic beam [3, 4] by solving the Maxwell equation by Fourier Transformation, the longitudinal impedance is showed as Eq. (7). And when we get the expression of the impedance, we can obtain the Longitudinal Wake Field through Inverse Fourier Transformation.

$$W_0'(z,r) = \frac{1}{2\pi v}\int_{-\infty}^{\infty} Z_\parallel(\omega,r)\,e^{\frac{i\omega z}{v}}d\omega \tag{6}$$

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It should be noticed that there is a coefficient named v on the denominator before the integration. This is because we should take the correspond Fourier Transformation pair when we do this operation.

Usually we take the numerical integration to get the Inverse Fourier Transformation for the longitudinal Impedance. As a benchmark, this relativistic formula is more general than the ultra case. And when we set the energy of the electron to infinity, it should approach to the ultra-relativistic limit naturally. Fig. 1 show that when we take the limit we said, the wake field have the feature that when $z > 0$, wake field is zero, when $z < 0$ it is not zero. This is the result of causality, which means that the speed of the electromagnetic field's propagation is not faster than the speed of the electron which is c when energy is infinity.

$$Z_{\parallel}(\omega, r) = \frac{iZ_0 c k_r^2}{2\pi\omega}\left[K_0(k_r r) + I_0(k_r r)\frac{\omega^2\lambda K_1(bk_r)K_0(b\lambda) + k_r c^2(\lambda^2 - k^2)K_0(bk_r)K_1(\lambda b)}{\omega^2\lambda I_1(bk_r)K_0(b\lambda) - k_r c^2(\lambda^2 - k^2)I_0(bk_r)K_1(\lambda b)}\right] \quad (7)$$

It should be noticed that in the mean time, at the long range, the wake field we calculate has a good agreement with the classical equation given by A.Chao [1], which is

$$W_0'(z) = -\frac{c}{4\pi b}\sqrt{\frac{Z_0}{\pi\sigma_c}}\frac{L}{|z|^{\frac{3}{2}}} \quad (8)$$

And an example is showed at Table 2 and Fig. 2

Table 2: Consistency Between Classical Equation and General Equation

z/m	χ	**formula**	**numerical**
-0.01	5.7×10^{-5}	-5.47×10^{9}	-5.35×10^{9}

TRANSVERSE WAKEFIELD

Misleading Point in Pervious Result

In essays [3,4] corresponding this question, they did not mention the mathematical expression for such a *dipole current*, the model they are considering about is pretty much vague. If we take the model they consider, which is

$$\begin{aligned} j_\varphi &= 0, \\ j_r &= 0, \\ \int j_s dr d\varphi &= v\lambda_b \end{aligned} \quad (9)$$

when we follow the derivation given by pervious result, we will have to determine a coefficient given by the strength of the dipole charge, and the equivalence mathematical problem is that we solve the green function given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\widetilde{A_s}}{\partial r}\right) - \left(\frac{1^2}{r^2} + k_r^2\right)\widetilde{A_s} = -\mu_0\widetilde{j_s} \quad (10)$$

The general solution of the problem is $CI_1(k_r r) + DK_1(k_r r)$. We can then expand the strange part of the general solution as

$$\begin{aligned} K_1(x) &= \frac{1}{x} + \frac{1}{4}x(2\ln x + 2\gamma - 1 - 2\ln 2) + \\ &\quad \frac{1}{64}x^3(4\ln x + 4\gamma - 5 - 4\ln 2) + O(x^4) \\ K_1(k_r r) &\approx \frac{1}{k_r r} \end{aligned} \quad (11)$$

this model will lead to a equation showed below

$$\Phi(0,0) = \lim_{\varepsilon\to 0} 2\pi\left(\Phi(x^*, y^*)D\frac{-1}{k_r\varepsilon} - D\frac{1}{k_r}\frac{\partial\Phi}{\partial r}(x^*, y^*)\right) \quad (12)$$

which can not determine the coefficient D in the general solution. So, the model given by pervious is not self-consistent.

Dipole Ring Model

We can give a *dipole ring model* and solve the electromagnetic field surround it. We explicitly give the source term of the dipole ring, which is Eq. (13)

$$\begin{aligned} \rho_m &= \frac{q}{\pi a}\delta(s - vt)\delta(r - a)\cos m\theta \\ j_m &= c\rho_m\hat{s} \end{aligned} \quad (13)$$

By solving the Maxwell equation under Lorentz Gauge, we give the general solution of vector potential A_r, A_φ, A_s and potential ϕ. In different area, we have

$$\begin{aligned} & r < a \\ A_r &= \frac{1}{2}\left(p_+^c I_2(k_r r) + p_-^c I_0(k_r r)\right)\cos\varphi e^{ikz} \\ A_\varphi &= \frac{1}{2}\left(p_+^c I_2(k_r r) - p_-^c I_0(k_r r)\right)\sin\varphi e^{ikz} \\ A_s &= p_s^c I_1(k_r r)\cos\varphi e^{ikz} \\ \phi &= p_0^c I_1(k_r r)\cos\varphi e^{ikz} \end{aligned} \quad (14)$$

where Lorentz Gauge requires that

$$p_+^c = -p_-^c, \quad p_0^c = \frac{c^2 k}{\omega}p_s^c \quad (15)$$

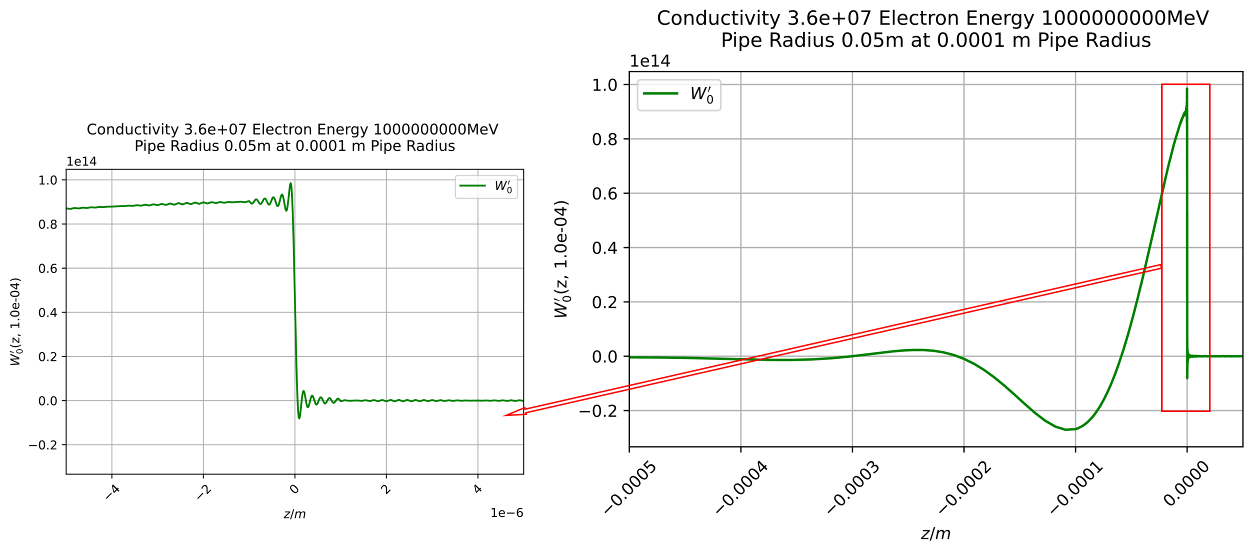

Figure 1: Wake field for ultra-relativistic electron near $r = 0$ as a benchmark.

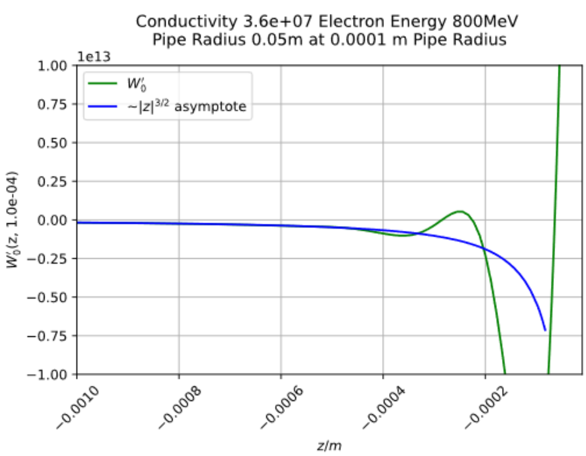

Figure 2: Long range asymptote.

then

$a < r < b$

$$
\begin{aligned}
A_r &= \frac{1}{2}\left(p_+ I_2\left(k_r r\right) + q_+ K_2\left(k_r r\right) + p_- I_0\left(k_r r\right)\right. \\
&\quad \left. + q_- K_0\left(k_r r\right)\right) \cos\varphi e^{ikz} \\
A_\varphi &= \frac{1}{2}\left(p_+ I_2\left(k_r r\right) + q_+ K_2\left(k_r r\right) - p_- I_0\left(k_r r\right)\right. \\
&\quad \left. - q_- K_0\left(k_r r\right)\right) \sin\varphi e^{ikz} \\
A_s &= \left(p_s I_1\left(k_r r\right) + q_s K_1\left(k_r r\right)\right) \cos\varphi e^{ikz} \\
\phi &= \left(p_0 I_1\left(k_r r\right) + q_0 K_1\left(k_r r\right)\right) \cos\varphi e^{ikz}
\end{aligned}
\tag{16}
$$

where Lorentz Gauge requires that

$$
p_- = -p_+, \quad p_0 = \frac{c^2 k}{\omega} p_s, \quad q_- = -q_+, \quad q_0 = \frac{c^2 k}{\omega} q_s \tag{17}
$$

The area $r > b$ are not listed. At last we can determine totally eight parameters through the continuous of electromagnetic field, which can be calculated by Mathematica. However, it is too long to show in this paper.

CONCLUSION

We finally get a more self-consistent dipole wake and benchmark the monopole wake. After this work, we can move forward to dynamic analysis.

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REFERENCES

[1] A. W. Chao, *Lectures on Accelerator Physics*, World Scientific, 2020. doi:10.1142/12004

[2] K. L. F. Bane and M. Sands, "The Short-Range Resistive Wall Wakefields," *AIP Conference Proceedings*, vol. 367, no. 1, pp. 131–149, 04 1996. doi:10.1063/1.50300

[3] G. Stupakov, "Resistive-wall wake for nonrelativistic beams revisited," *Phys. Rev. Accel. Beams*, vol. 23, p. 094401, Sep 2020. doi:10.1103/PhysRevAccelBeams.23.094401

[4] F. Zimmermann and K. Oide, "Resistive-wall wake and impedance for nonultrarelativistic beams," *Phys. Rev. ST Accel. Beams*, vol. 7, p. 044201, Apr 2004. doi:10.1103/PhysRevSTAB.7.044201