Extreme Light Infrastructure-Nuclear Physics
(ELI-NP) - Phase II

# Solitary and Shock Waves in Free and Magnetized Quasi-neutral Laser Induced Plasmas


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- Basic Properties of Laser Plasmas
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# Brief History of plasma acceleration

- **In 1956, Veksler and Budker proposed using plasma collective fields to accelerate charged particles more compactly.**
- In 1979, Tajima and Dawson showed how an ***intense laser pulse can excite a wake of plasma oscillations*** through the **non-linear ponderomotive force associated to the laser pulse.**
- In 1994, at the RAL, using the **40 TW powerful Vulcan Laser**, **hundreds of GV/m gradients** have been generated and used to accelerate electrons to **few tens of MV/m over only 1 mm** distance.
- In 1985, Chen and Dawson proposed to use a **bunched electron beam to drive plasma wakes with GV/m accelerating gradients**.
- In 2009, Caldwell, Lotov, Pukhov and F. Simon proposed to drive plasma-wakefield acceleration **with a proton bunch**, and the authors **demonstrated numerically that TeV energy levels could be reached in a single accelerating stage driven by a TeV proton bunch**.

# Basic Properties of Laser Plasmas

To gain insight into the basic laws and properties of laser induced plasmas, consider the following **simple non-relativistic model**

$$\partial_t n + \partial_x (n v_x) = 0$$

$$\partial_t \mathbf{v} + v_x \partial_x \mathbf{v} = -\frac{e}{m}[\mathbf{E} + \mathbf{e}_x(\mathbf{v}\cdot\partial_x \mathbf{A}) - v_x \partial_x \mathbf{A}]$$

$$\partial_x E_x = -\frac{e}{\epsilon_0}(n - \bar{Z} n_i)$$

This model describes the **plasma response** to an ***external perturbation*** propagating longitudinally along the *x*-axis and specified by the **electromagnetic vector potential A**.

Neglecting the time variation of the ***transverse components*** $\mathbf{v}_\perp$ of the current velocity, it follows that

$$\mathbf{v}_\perp = \frac{e\mathbf{A}_\perp}{m}$$

# Basic Properties of Laser Plasmas Continued

Thus, the momentum balance equation can be rewritten as

$$\partial_t v_x + \frac{1}{2}\partial_x v_x^2 = -\frac{eE_x}{m} - \frac{e^2}{2m^2}\partial_x A^2$$

**Ponderomotive force**

Linearize the continuity equation, the Poisson equation and the one above with $\bar{Z}n_i = n_0$ and $n = n_0 + \tilde{n}$. Applying the traveling wave approximation by introducing a new variable $\xi = x - ut$, we obtain

$$\left(\partial_\xi^2 + k_e^2\right)\tilde{n} = \frac{\epsilon_0 k_e^2}{2m}\partial_\xi^2 A^2$$

**Harmonic oscillation of the electron density modulation driven by the gradient of the ponderomotive force.**

$$k_e = \frac{\omega_e}{u}$$

$$\omega_e = \sqrt{\frac{e^2 n_0}{m\epsilon_0}}$$

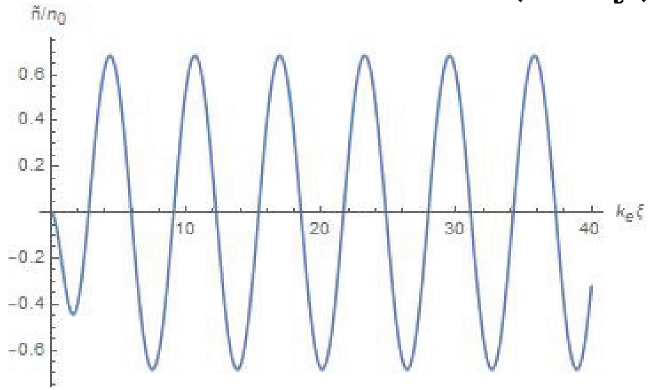
**Electron plasma frequency**

# Basic Properties of Laser Plasmas Continued

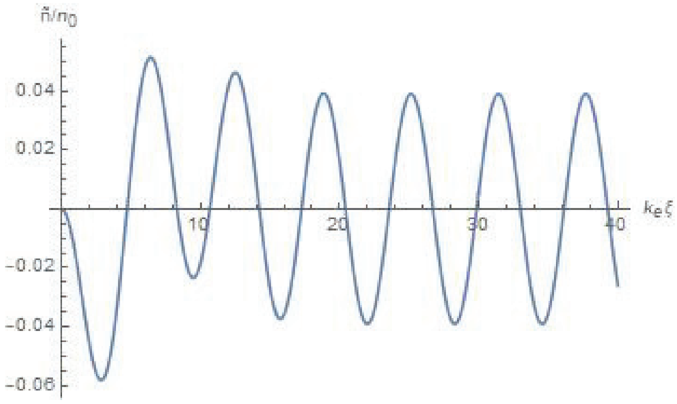
For sufficiently short driving laser pulse of the form

$$A^2 = \left(\frac{mca_0}{e}\right)^2 \exp\left(-\frac{\xi^2}{\sigma_l^2}\right)$$

$$a_0 \approx 0.855\lambda[\mu m]\sqrt{I[10^{18}W cm^{-2}]}$$




The driving laser pulse is resonant for $\boldsymbol{k_e\sigma_l = \sqrt{2}}$. Here $\boldsymbol{a_0 = 1.3}$ $\boldsymbol{\omega_e \sim 1.8 \times 10^{11}}$ **Hz**




Electron density evolution for a non resonant value of $\boldsymbol{k_e\sigma_l = 5\sqrt{2}}$.

# Basic Properties of Laser Plasmas Continued

Apply now the traveling wave approximation to all basic equations. Manipulating the result, we obtain a single equation for the scalar potential $\left(E_x = -\partial_\xi \varphi\right)$

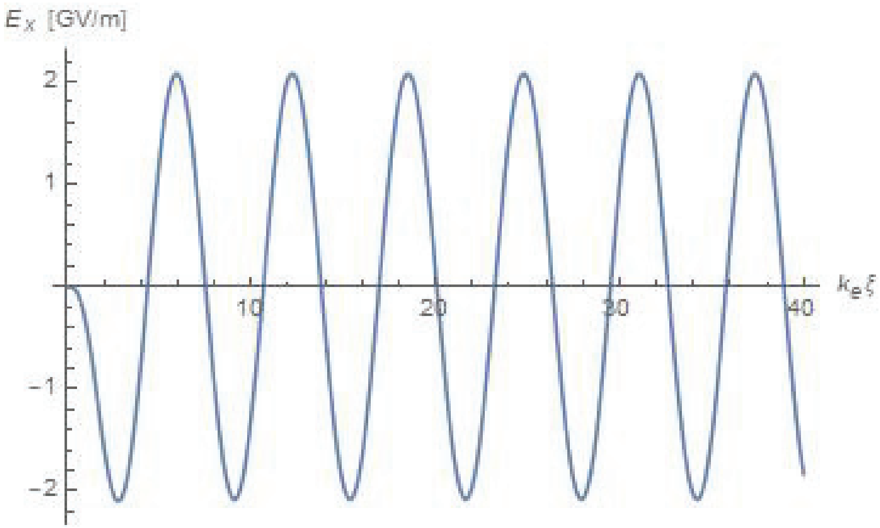
$$\partial_\xi^2 \varphi = \frac{en_0}{\epsilon_0}\left\{\left[1 + \frac{2e\varphi}{mu^2} - \frac{e^2}{m^2u^2}\left(A^2 - A_0^2\right)\right]^{-1/2} - 1\right\}$$

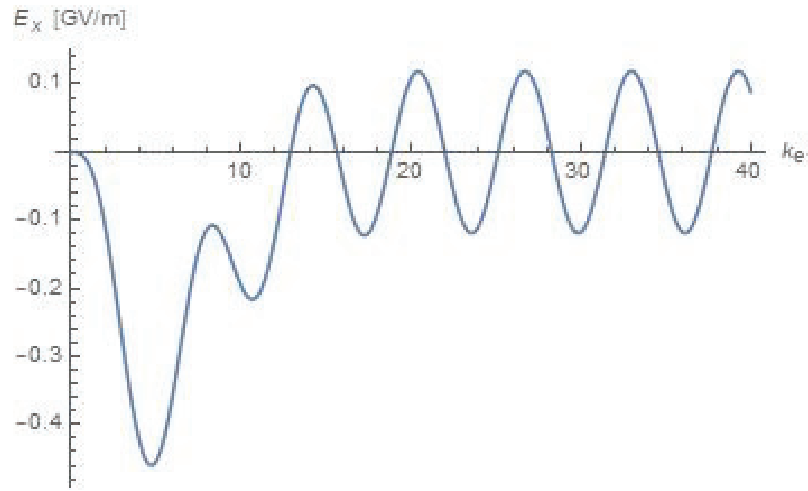
Expanding the square root on the right-hand-side, we obtain

$$\left(\partial_\xi^2 + k_e^2\right)E_x = -\frac{ek_e^2}{2m}\partial_\xi A^2$$

# Basic Properties of Laser Plasmas Continued

Evolution of the longitudinal electric field $E_x$ for the resonant and the non-resonant case, where $\boldsymbol{a_0 = 1.3}$ and $\boldsymbol{\omega_e \sim 1.8}$ **THz**




For typical plasma number densities of the order of $\boldsymbol{10^{21}\ m^{-3}}$, **the impressive acceleration gradients of the order of several gigavolts per meter can be reached**.

**Quasi-neutral plasma** comprised of electrons and ions in an **external electromagnetic field** depending on the ***scaled coordinates*** $x = (x, y, s)$ and the ***scaled dimensionless time*** $t$.

Nonlinear Vlasov equation

$$\partial_t f_a + \frac{\mathbf{p}_\perp - Z_a \mathbf{A}_\perp}{\mu_a \gamma_a} \cdot \nabla_\perp f_a + \frac{p_s}{\mu_a \gamma_a} \partial_s f_a + (Z_a \mathcal{F} - \mu_a \partial_s \gamma_a) \partial_{p_s} f_a = 0$$

Here $\mu_a = {}^{m_a}/_{m} \Longrightarrow$ **mass aspect ratio** with respect to the electron mass.

**Electric force** $\Longrightarrow$ $\mathcal{F} = -\partial_s \Phi - \partial_t A_s$

**Spatial variations are one-dimensional** in nature, so that the partial derivatives $\partial_x = \partial_y = 0$, while $\partial_s$ is generally nonzero.

**Transverse canonical momenta $\mathbf{p}_\perp$ are integrals of motion**.

The full Vlasov equation is then reduced to the **one-dimensional**

$$\partial_t F_a + \frac{p_s}{\mu_a \gamma_a} \partial_s F_a + (Z_a \mathcal{F} - \mu_a \partial_s \gamma_a) \partial_{p_s} F_a = 0$$

where now $\gamma_a(s, p_s; t) = \sqrt{1 + \frac{1}{\mu_a^2} [p_s^2 + Z_a^2 A^2(s;t)]}$ and $A^2 = A_x^2 + A_y^2$

Consider a class of **water bag distributions** solving exactly the one-dimensional Vlasov equation, which are given by

$$F_a(s, p_s; t) = \mathcal{C}_a \left\{ \Theta \left[ p_s - p_a^{(-)}(s;t) \right] - \Theta \left[ p_s - p_a^{(+)}(s;t) \right] \right\}$$

We introduce an ***important quantity*** $\Gamma_a$, which can be written as

$$\Gamma_a = \left[ \frac{1}{(1 - V_a^2)(1 - 2 v_{aT}^2 n_a^2)} \left( 1 + \frac{Z_a^2}{\mu_a^2} A^2 \right) \right]^{1/2}$$

**Remarkable Lorentz invariant EXACT hydrodynamic closure** of macroscopic equations for each plasma species coupled with the wave equations for the self-consistent fields

$$\partial_t(n_a\Gamma_a) + \partial_s(n_a\Gamma_a V_a) = 0$$

$$\partial_t(V_a\Gamma_a) + \partial_s\Gamma_a = \mathcal{F}_a = -\frac{Z_a}{\mu_a}(\partial_s\Phi + \partial_t A_s)$$

$$\Box\Phi = -\frac{1}{n_{e0}}\sum_a Z_a n_a \Gamma_a$$

$$\Box A_s = -\frac{1}{n_{e0}}\sum_a Z_a n_a \Gamma_a V_a$$

$$\Box \mathrm{A}_\perp = \frac{\mathrm{A}_\perp}{n_{e0}}\sum_a Z_a^2 n_a\left(1 + \frac{2}{3}v_{aT}^2 n_a^2\right) + \Box \mathrm{A}_e$$

**Ions comprise a heavy plasma background**, so that their effect on the formation and the dynamics of the plasma wakefield is neglected. Using the Lorentz gauge $\partial_t \Phi + \nabla \cdot \mathbf{A} = 0$ we have

$$\Box \mathcal{F} = \partial_s (n\Gamma) + \partial_t (n\Gamma V)$$

Thus, ***instead of the hydrodynamic closure***, the **basic equations** to be analysed become

$$\boldsymbol{\partial_t (n\Gamma) + \partial_s (n\Gamma \mathbf{V}) = 0}$$

$$\boldsymbol{\Box\left[\partial_t^2 (\Gamma \mathbf{V}) + \partial_t \partial_s \Gamma\right] = -\Box (n\Gamma \mathbf{V})}$$

$$\boldsymbol{\Box \mathbf{A}_\perp = n\left(1 + \frac{2}{3} v_T^2 n^2\right) \mathbf{A}_\perp}$$

# Continued NONLINEAR WAVES AND COHERENT STRUCTURES IN QUASI-NEUTRAL PLASMAS

- Standard procedure of the **multiple scales reduction method**.
- The electron density $n$, the current velocity $V$ and the transverse vector potential $\mathbf{A}_\perp$ ***are expanded in a formal small expansion parameter***.
- Then, the corresponding perturbation equations are solved, such that **secular terms are eliminated order by order**.
- As a result, the evolution dynamics of the hydrodynamic and field variables is being split on different spatial and time scales – **fast ones** involving rapid wave oscillations and **slow scales** on which coherent motion of certain wave amplitudes occurs.
- For the case of **constant phase-space density distribution**, the macroscopic fluid description is ***fully equivalent*** to the nonlinear Vlasov-Maxwell equations and the corresponding wave equations for the self fields.

# Continued NONLINEAR WAVES AND COHERENT STRUCTURES IN QUASI-NEUTRAL PLASMAS

Hydrodynamic and field variable can be written as

$$n(s;t) = 1 + \frac{k}{G_0^2\Omega}\left[\mathcal{B}(s;t)e^{i\varphi(s;t)} + \mathcal{B}^*(s;t)e^{-i\varphi(s;t)}\right]$$

$$V(s;t) = \mathcal{B}(s;t)e^{i\varphi(s;t)} + \mathcal{B}^*(s;t)e^{-i\varphi(s;t)}$$

$$A_\perp(s;t) = \mathcal{A}(s;t)e^{i\psi(s;t)} + \mathcal{A}^*(s;t)e^{-i\psi(s;t)}$$

where $G_0 = (1 - 2v_T^2)^{-1/2}$

$$\varphi = ks - \Omega t \quad \psi = ks - \omega t$$

$$\Omega = \sqrt{1 + 2k^2v_T^2} \qquad \omega = \sqrt{1 + k^2 + \frac{2}{3}v_T^2}$$

## Key result

$$i\partial_t \boldsymbol{\mathcal{A}} + i v_\omega \partial_s \boldsymbol{\mathcal{A}} = -\frac{1}{2}\frac{dv_\omega}{dk}\partial_s^2 \boldsymbol{\mathcal{A}} + \Gamma_{aa}\mathcal{A}^2 \boldsymbol{\mathcal{A}}^* + \Gamma_{ab}|\mathcal{B}|^2 \boldsymbol{\mathcal{A}}$$

$$i\partial_t \mathcal{B} + i v_\Omega \partial_s \mathcal{B} = -\frac{1}{2}\frac{dv_\Omega}{dk}\partial_s^2 \mathcal{B} + \Gamma_{ba}|\mathcal{A}|^2 \mathcal{B} + \Gamma_{bb}|\mathcal{B}|^2 \mathcal{B}$$

$\mathcal{A}^2 = \boldsymbol{\mathcal{A}} \cdot \boldsymbol{\mathcal{A}}$ is **complex**, while $|\mathcal{A}|^2 = \boldsymbol{\mathcal{A}} \cdot \boldsymbol{\mathcal{A}}^*$ is **real**.

**The above equations comprise a system of a nonlinear vector Schrodinger equations for $\mathcal{A}$ coupled to a scalar nonlinear Schrodinger equation for $\mathcal{B}$. They describe the evolution of the slowly varying amplitudes of the generated transverse plasma wakefield and the current velocity of the plasma electrons**.

# Continued NONLINEAR WAVES AND COHERENT STRUCTURES IN QUASI-NEUTRAL PLASMAS

Relation between the components of $\boldsymbol{\mathcal{A}} \Longrightarrow \mathcal{A}_y = \mathcal{C}\mathcal{A}_x$

- $\mathcal{C} = p$ is real. The incident $p = 0$ corresponds to the case of **linear wave polarization**.
- $\mathcal{C} = \pm i$. This corresponds to **circular wave polarization**.

**We shall analyse circularly polarized plasma waves**

$$i\partial_t\mathcal{A}_x + iv_\omega\partial_s\mathcal{A}_x = -\frac{1}{2}\frac{dv_\omega}{dk}\partial_s^2\mathcal{A}_x + \Gamma_{ab}|\mathcal{B}|^2\mathcal{A}_x$$

$$i\partial_t\mathcal{B} + iv_\Omega\partial_s\mathcal{B} = -\frac{1}{2}\frac{dv_\Omega}{dk}\partial_s^2\mathcal{B} + 2\Gamma_{ba}|\mathcal{A}_x|^2\mathcal{B} + \Gamma_{bb}|\mathcal{B}|^2\mathcal{B}$$

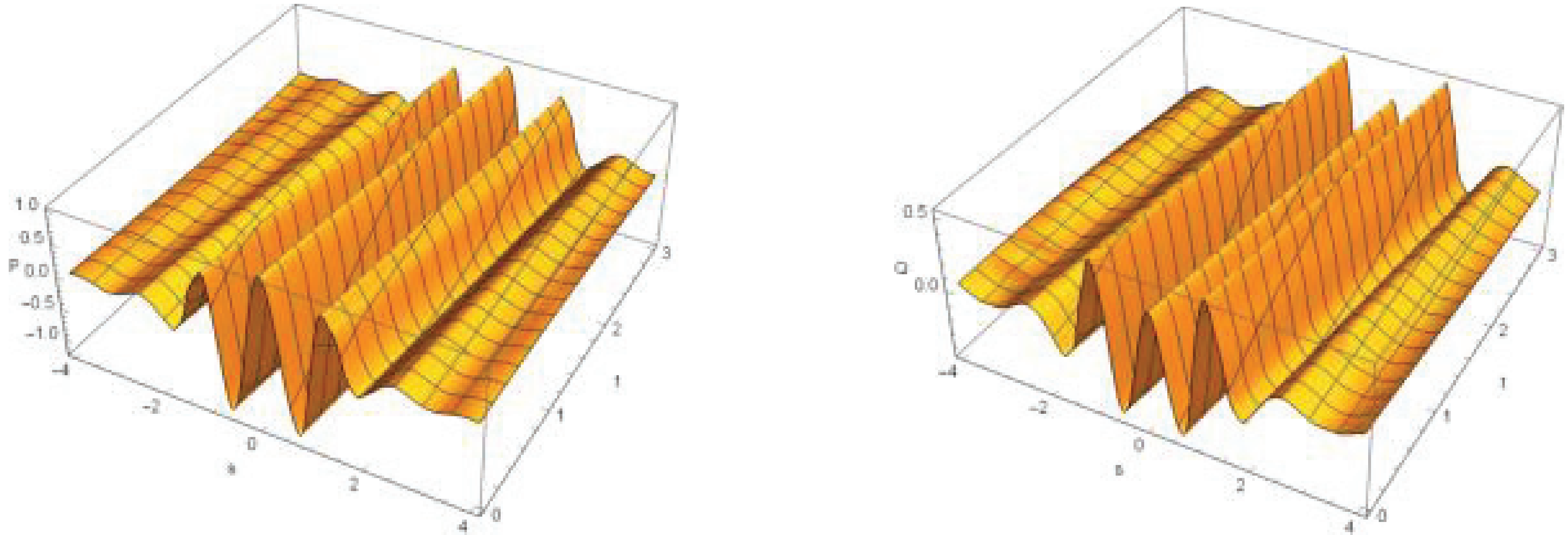
We shall describe **traveling wave solutions** through the ansatz

$$\mathcal{A}_x = e^{i(\mu\xi+\nu_1\eta)}\mathcal{P}(\eta - u\xi), \qquad \mathcal{B} = e^{i(\mu\xi+\nu_2\eta)}\mathcal{Q}(\eta - u\xi)$$

$$\xi = \tilde{a}(s - v_\omega t), \qquad \eta = -\tilde{a}(s - v_\Omega t), \qquad \tilde{a} = (v_\Omega - v_\omega)^{-1}$$

# Continued NONLINEAR WAVES AND COHERENT STRUCTURES IN QUASI-NEUTRAL PLASMAS

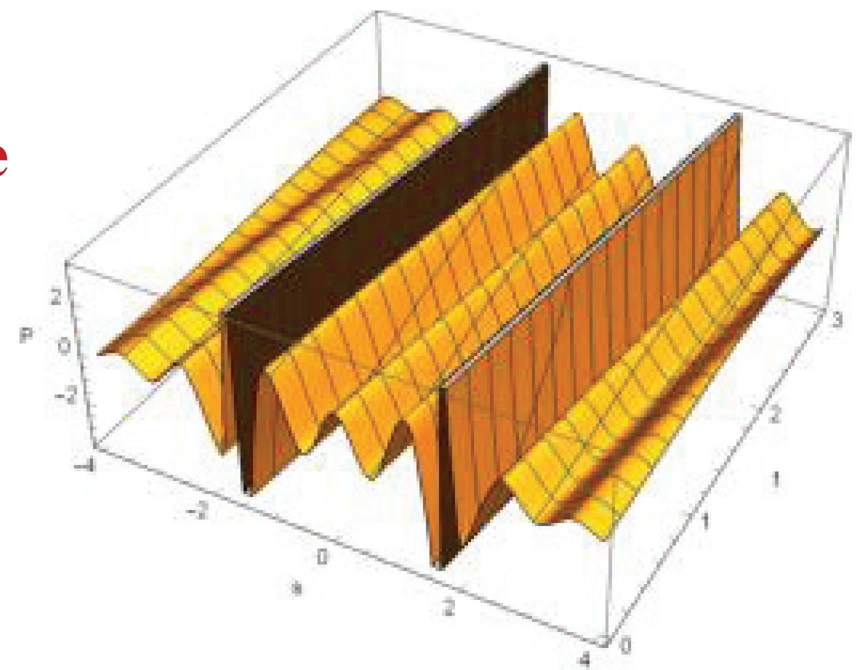
The resulting equations for $\mathcal{P}$ and $\mathcal{Q}$ have been solved by the **method of formal series of Dubois-Violette**. Their solution is represented by a **ratio of two formal Volterra series**. It is compact and elegant and very useful for practical applications.




**Evolution of the traveling wave amplitudes $\mathcal{P}$ (left) and $\mathcal{Q}$ for the case k = 1.543613, $v_T^2 = 0.1$ and $\mu = 1.0$.**

A **remarkable property** of the formal series solution is the fact that **near a resonance the denominator is divergent at least as much as the numerator**, so that their ratio gives a reasonable and relevant for applications result. This property is demonstrated in the figure.

**Evolution of the traveling wave amplitude $\mathcal{P}$ close to a linear resonance $\omega_1 - \omega_2 = 0$. The values of the corresponding parameters are $k = 1.543613$, $v_T^2 = 0.1$ and $\mu = 2.0245$.**



We analyse the properties of a plasma comprised of electrons and ions immersed in an external constant magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_x$. The dimensionless **Vlasov-Maxwell system** of equations is

$$\partial_t f_a + \mathbf{v}\cdot\nabla f_a - \nu_a \mathbf{e}_x \times \mathbf{v}\cdot\nabla_p f_a + Z_a(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_p f_a = 0$$

$$\Box\mathbf{A} = -\sum_a \lambda_a \int d^3\mathbf{p}\,\mathbf{v} f_a(\mathbf{x},\mathbf{p};t)$$

$$\Box\varphi = -\sum_a \lambda_a \int d^3\mathbf{p}\, f_a(\mathbf{x},\mathbf{p};t)$$

$$\partial_t\mathbf{E} = \nabla(\nabla\cdot\mathbf{A}) - \partial_t^2\mathbf{A} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\lambda_a = \frac{Z_a n_a}{n_e} \qquad \nu_a = \frac{\mu_a\omega_a}{\omega_e} \qquad \omega_a = \frac{q_a B_0}{m_a}$$

## Hydrodynamic substitution

$$f_a(\mathbf{x},\mathbf{p};t) = \varrho_a(\mathbf{x};t)\boldsymbol{\delta}^3\left[\mathbf{p} - \frac{1}{\mu_a}\gamma_a(\mathbf{x};t)\mathbf{v}_a(\mathbf{x};t)\right]$$

Substitution in Vlasov equation leads to *cold hydrodynamics pic*

$$\partial_t\varrho_a + \nabla\cdot(\varrho_a\mathbf{v}_a) = 0$$

$$\partial_t(\gamma_a\mathbf{v}_a) + \mathbf{v}_a\cdot\nabla(\gamma_a\mathbf{v}_a) + \bar{\omega}_a\mathbf{e}_x\times\mathbf{v}_a = \mu_a Z_a(\mathbf{E}+\mathbf{v}_a\times\mathbf{B})$$

$$\Box\mathbf{A} = -\sum_a \lambda_a\varrho_a\mathbf{v}_a \qquad \Box\varphi = -\sum_a \lambda_a\varrho_a$$

$$\bar{\omega}_a = \frac{\omega_a}{\omega_e}$$

$\omega_e$ is the **electron plasma frequency**.

# Cont-ed NONLINEAR WAVES AND COHERENT STRUCTURES IN MAGNETIZED PLASMAS

A special case of **plasma wave anisotropy**, implying that the longitudinal and the transverse plasma waves ***depend on the longitudinal*** (in the direction of the applied external magnetic field $\mathbf{B}_0$) $x$ coordinate only. **This assumption is not essential**, however it simplifies the analytical treatment considerably. Simplified **notations and new complex valued variables**

$$v = v_x \qquad V = v_y + i v_z \qquad E = E_x$$

$$\mathcal{E} = E_y + i E_z \qquad A = A_x \qquad \mathcal{A} = A_y + i A_z$$

$$B_x = 0 \qquad \mathcal{B} = B_y + i B_z = i \partial_x \mathcal{A}$$

# Cont-ed NONLINEAR WAVES AND COHERENT STRUCTURES IN MAGNETIZED PLASMAS

The **basic equations** to be analysed can be written as

$$\partial_t \varrho + \partial_x (\varrho v) = 0$$

$$\partial_t (\gamma v) + v \partial_x (\gamma v) = -E - \frac{1}{2} \left( V \partial_x \mathcal{A}^* + V^* \partial_x \mathcal{A} \right)$$

$$\partial_t (\gamma V) + i \bar{\omega}_e V + v \partial_x (\gamma V) = \partial_t \mathcal{A} + v \partial_x \mathcal{A}$$

$$\Box A = \varrho v \qquad \Box \mathcal{A} = \varrho V \qquad \partial_t E = \Box A$$

**Linearize** the above equations about the **stationary solution** $\varrho_s = 1$ and manipulate the resulting equations in an obvious manner.

# Cont-ed NONLINEAR WAVES AND COHERENT STRUCTURES IN MAGNETIZED PLASMAS

$$\Box(\partial_t^2 + 1)A_1 = 0 \qquad \left(\Box\hat{\mathcal{L}} - \partial_t\right)\mathcal{A}_1 = 0 \qquad \hat{\mathcal{L}} = \partial_t + i\overline{\omega}_e$$

$A_1$ and $\mathcal{A}_1$ are the **linear parts** of the vector potential.
Solution of the first equation $A_1 = \mathcal{B}e^{it} + \mathcal{B}^* e^{-it}$ $\Longrightarrow$ **electrostatic plasma oscillations**. These **do not couple** to the whistler modes so one can set $\mathcal{B} = 0$.
The second equation yields the **dispersion relation for the transverse whistler waves**

$$D(k,\omega) = \omega - \Box_\omega \mathcal{L}_\omega \qquad \Box_\omega = \omega^2 - k^2 \qquad \mathcal{L}_\omega = \omega - \overline{\omega}_e$$

It can be easily verified that for ***typical values of the electron cyclotron frequency*** $\overline{\omega}_e$ the dispersion equation possesses **three distinct real roots**.

The transverse part of the electromagnetic vector potential reads

$$\mathcal{A}(x;t) = \sum_{n=1}^{3} \mathcal{C}_n(x;t) e^{i\psi_n(x;t)} \qquad \psi_n = kx - \omega_n t$$

The amplitudes $\mathcal{C}_n$ satisfy the following system of three coupled nonlinear Schrodinger equations

$$i\partial_t \mathcal{C}_n + i v_{gn} \partial_x \mathcal{C}_n = -\frac{1}{2}\frac{dv_{gn}}{dk}\partial_x^2 \mathcal{C}_n + \sum_m \Pi_{mn} \mathcal{C}_n |\mathcal{C}_m|^2 + \sum_{m \neq n} \Gamma_{mn} \mathcal{C}_n |\mathcal{C}_m|^2$$

Transverse components of the current velocity

$$V = \sum_{n=1}^{3} \square_n \mathcal{C}_n e^{i\psi_n}$$

# Cont-ed NONLINEAR WAVES AND COHERENT STRUCTURES IN MAGNETIZED PLASMAS

- **System of three coupled nonlinear Schrodinger equations** for the envelopes $C_n$ of the three whistler wave modes ***determined by the roots of the dispersion equation***.
- **Terms with *m* = *n*** are **excluded from the second sum** on the right-hand-side.
- This implies that the matrix of coupling coefficients $\Gamma_{mn}$ **represents a sort of a selection rule, according to which a generic mode *n* cannot couple with itself**.
- This feature is a consequence of the ***vector character of the nonlinear coupling between modes and is due to the nonlinear Lorentz force***.
- The first term (not present in the non relativistic case) involving the coupling matrix $\Pi_{mn}$ **allows self-coupling and is entirely due to the relativistic character of the motion**.

# Cont-ed NONLINEAR WAVES AND COHERENT STRUCTURES IN MAGNETIZED PLASMAS

Straightforward evaluation of the dispersion coefficients $v'_{gn} = dv_{gn}/dk$ shows that in a relatively wide range of plasma parameters one of them, say $v'_{gn}$ is several orders of magnitude smaller than the other two. Thus, in a good approximation

$$i\partial_t C_1 + iv_{g1}\partial_x C_1 = -\frac{v'_{g1}}{2}\partial_x^2 C_1 + \left(\Pi_{11}|C_1|^2 + \Sigma_{21}|C_2|^2 + \Sigma_{31}|C_3|^2\right)C_1$$

$$i\partial_t C_2 + iv_{g2}\partial_x C_2 = \left(\Sigma_{12}|C_1|^2 + \Pi_{22}|C_2|^2 + \Sigma_{32}|C_3|^2\right)C_2$$

$$i\partial_t C_3 + iv_{g3}\partial_x C_3 = -\frac{v'_{g3}}{2}\partial_x^2 C_3 + \left(\Sigma_{13}|C_1|^2 + \Sigma_{23}|C_2|^2 + \Pi_{33}|C_3|^2\right)C_3$$

**The equation for $\mathcal{C}_2$ possesses a simple solution of the form**

$\mathcal{C}_2 = g_2 e^{-i\Psi(x;t)}$ with $g_2 = \mathrm{const}$. The phase $\Psi$ satisfies

$$\partial_t \Psi + v_{g2} \partial_x \Psi = \Sigma_{12}|\mathcal{C}_1|^2 + \Pi_{22} g_2^2 + \Sigma_{32}|\mathcal{C}_3|^2$$

$$\Sigma_{mn} = \Pi_{mn} + \Gamma_{mn} \qquad\qquad m \neq n$$

This implies that our initial system can be reduced to a simpler system of **two coupled nonlinear Schrodinger equations**

$$i\partial_t \mathcal{C}_1 + i v_{g1} \partial_x \mathcal{C}_1 = -\frac{v'_{g1}}{2}\partial_x^2 \mathcal{C}_1 + \left(\Pi_{11}|\mathcal{C}_1|^2 + \Sigma_{21} g_2^2 + \Sigma_{31}|\mathcal{C}_3|^2\right)\mathcal{C}_1$$

$$i\partial_t \mathcal{C}_3 + i v_{g3} \partial_x \mathcal{C}_3 = -\frac{v'_{g3}}{2}\partial_x^2 \mathcal{C}_3 + \left(\Sigma_{13}|\mathcal{C}_1|^2 + \Sigma_{23} g_2^2 + \Pi_{33}|\mathcal{C}_3|^2\right)\mathcal{C}_3$$

# Cont-ed NONLINEAR WAVES AND COHERENT STRUCTURES IN MAGNETIZED PLASMAS

We shall describe **traveling wave solutions** through the ansatz

$$\mathcal{C}_1 = e^{i(\mu_1\xi+\mu_2\eta)}\mathcal{P}_1(\eta), \qquad \mathcal{C}_3 = e^{i\mu_3(\xi+\eta)}\mathcal{P}_3(\eta)$$

$$\xi = -\tilde{a}(x - v_{g1}t), \quad \eta = \tilde{a}(x - v_{g3}t), \quad \tilde{a} = (v_{g1} - v_{g3})^{-1}$$

The **characteristic frequencies**

$$\nu_1^2 = -\frac{2}{v'_{g1}\tilde{a}^2}\left(\mu_1 - \frac{1}{2v'_{g1}\tilde{a}^2} + g_2^2\Sigma_{21}\right)$$

$$\nu_3^2 = -\frac{2}{v'_{g3}\tilde{a}^2}(\mu_3 + g_2^2\Sigma_{23})$$

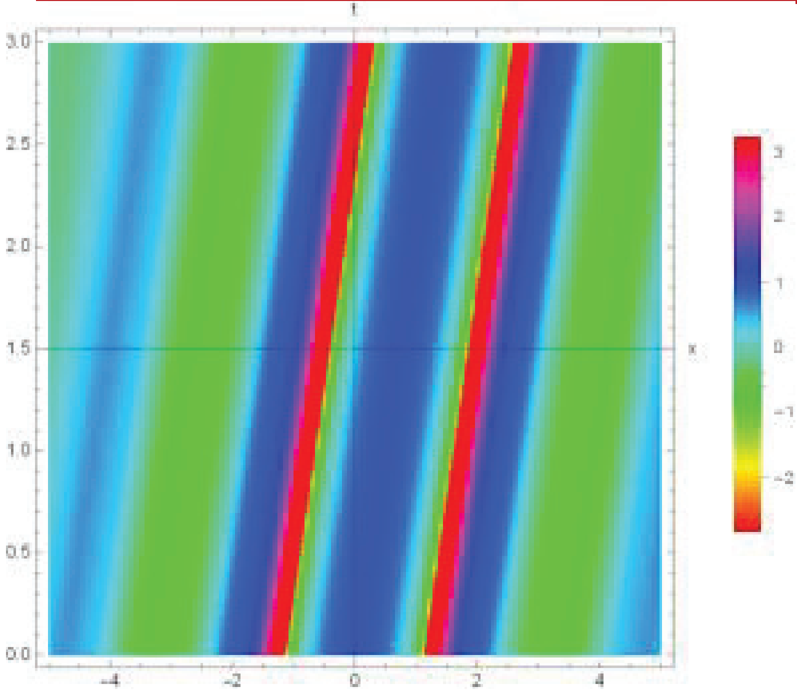
**are real**, provided

$$\mu_3 > 0 \qquad \mu_1 < \frac{1}{2v'_{g1}\tilde{a}^2} - g_2^2\Sigma_{21}$$

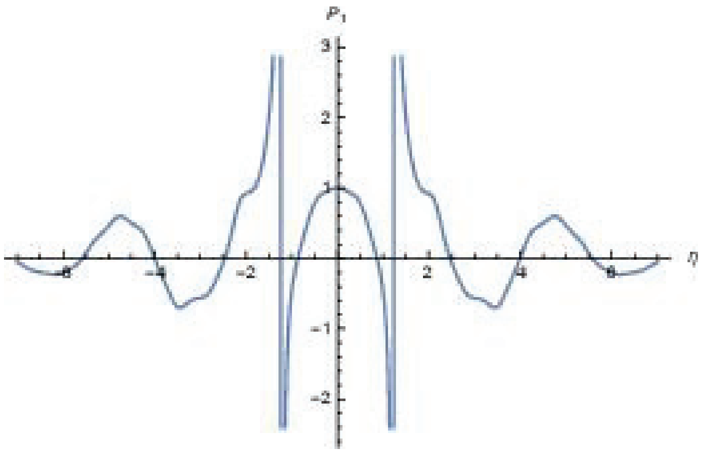
Here we describe this particular case.

# Cont-ed NONLINEAR WAVES AND COHERENT STRUCTURES IN MAGNETIZED PLASMAS

The resulting equations for $\mathcal{P}_1$ and $\mathcal{P}_3$ have been solved by the **method of formal series of Dubois-Violette**.

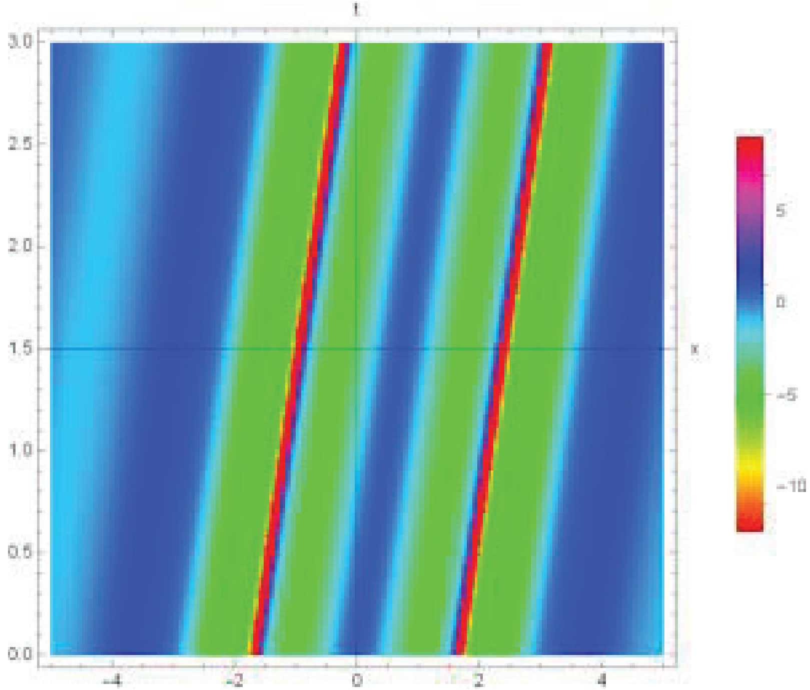


Evolution of the non relativistic traveling wave amplitude $\mathcal{P}_1$ for the case $\bar{\omega}_e = 1$ $k = 1$, $\mu_1 = -1$, $\mu_3 = 1$ and $g_2 = 0$.

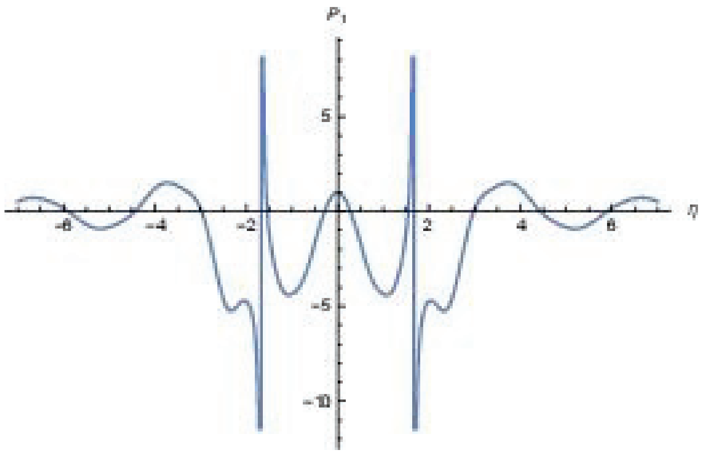


Evolution of the non relativistic traveling wave amplitude $\mathcal{P}_1$ as a function of $\eta$ for the case $\bar{\omega}_e = 1$, $k = 1$, $\mu_1 = -1$, $\mu_3 = 1$ and $g_2 = 0$.

## Relativistic case




Evolution of the relativistic traveling wave amplitude $\mathcal{P}_1$ for the case $\overline{\omega}_e = 1$ $k = 1$, $\mu_1 = -1$, $\mu_3 = 1$ and $g_2 = 0$.



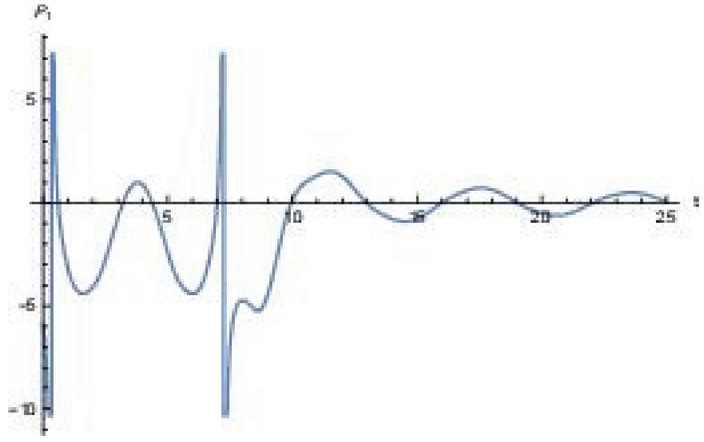

Evolution of the relativistic traveling wave amplitude $\mathcal{P}_1$ as a function of $\eta$ for the case $\overline{\omega}_e = 1$, $k = 1$, $\mu_1 = -1$, $\mu_3 = 1$ and $g_2 = 0$.

# Cont-ed NONLINEAR WAVES AND COHERENT STRUCTURES IN MAGNETIZED PLASMAS

- Traveling wave solution represents **$1/\eta$ -damped quasi-periodic oscillations** of the whistler amplitudes, which fade away with respect to the travelling wave variable $\eta$.
- The solitary-like wave crests (positive, as well as negative) with respect to the spatial variable for both $\mathcal{P}_1$ and $\mathcal{P}_3$ are almost monolithic structures, which are stable in time and are **symmetrically located on both sides of the line $x = v_{g3}t$**.
- The plasma response to the induced whistler waves consists in **transverse velocity redistribution**, which follows exactly the nonlinear behaviour of the whistlers. This means that the electron current flow is **well confined and localized in the transverse direction**, such that on a scale $3 \sim 4\ c/\omega_e$ the tails of the electron density distribution can be considered as ***practically completely subdued***.

# Cont-ed NONLINEAR WAVES AND COHERENT STRUCTURES IN MAGNETIZED PLASMAS

In both the **non relativistic and the fully relativistic case**, the whistler mode amplitudes $\mathcal{P}_1$ and $\mathcal{P}_3$ at a **fixed location in the longitudinal direction *x* decay rapidly in time**.




**Time evolution of the fully relativistic traveling wave amplitude $\mathcal{P}_1$ for a particular value $x = 1.835$ of the longitudinal coordinate *x*. Here $\bar{\omega}_e = 1$, $k = 1$, $\mu_1 = -1$, $\mu_3 = 1$ and $g_2 = 0$.**

# CONCLUDING REMARKS

- The **principle of generating super-strong electric** accelerating fields has been **demonstrated using a simple and illustrative physical model**.
- An exact **relativistic hydrodynamic closure of equations** describing the dynamics of various species in a quasi-neutral plasma has been obtained.
- The set of equations for the macroscopic hydrodynamic variables coupled to the wave equations for the self-consistent electromagnetic field is **fully equivalent to the Vlasov-Maxwell system for a special type of relativistic water-bag solutions** of the Vlasov equation.
- A system comprising a **vector nonlinear Schrodinger equation for the transverse envelopes of the self-consistent plasma wakefield coupled to a scalar nonlinear Schrodinger equation for the electron current velocity** envelope has been derived.
- Damped ***quasi-periodic traveling waves possess a solitary (shock) and multipeak structure*** and are possibly related to recent experiments on the so-called "shock acceleration“.

# CONCLUDING REMARKS

- Utilizing a technique known as **the hydrodynamic substitution, a relativistic hydrodynamic system of equations** describing the dynamics of various species in a cold quasi-neutral plasma immersed in an **external solenoidal magnetic field** has been obtained.
- **A system comprising three coupled nonlinear Schrodinger equation for the three basic whistler modes has been derived**.
- An intriguing feature of the description is that **whistler waves do not perturb the initial uniform density distribution** of plasma electrons. The plasma response to the induced whistler waves consists in **transverse velocity redistribution**, which follows exactly the behaviour of the whistlers.
- The electron current flow is **well localized in the transverse direction**, such that on a **spatial scale of 3 $\sim 4c/\omega_e$ the tails of the electron density distribution can be considered as practically completely faded away**.
- **This property may have an important application for transverse focusing of charged particle beams in future laser plasma accelerators**.

# The END