DEGENERATE SELF-CONSISTENT DISTRIBUTIONS FOR CHARGED PARTICLE BEAM IN LINEAR TRANSVERSE FIELD

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Abstract

The present report is devoted to covariant description of the microcanonical distributions for a charged particle beam on the base of the covariant approach previously developed in the works of the author.

For microcanonical distribution, particles are distributed on some ellipsoid, representing three-dimensional surface in the four-dimensional phase space.

Goal of this work is to analyze how do particles are distributed on the surface of this ellipsoid It means that we concerned with particle density on that surface. Besides, we investigate evolution of the density in time to verify that spatial two-dimensional density in projection of the ellipsoid onto configuration space remains uniform.

Our approach is distinct from previously used common approach, according to which microcanonical distribution is described by the delta-function, and particle distribution on the ellipsoid is not considered.

INTRODUCTION

Microcanonical distribution for charged particle beam was firstly introduced in 1959 by Soviet physicists Kapchinsky and Vladimirsky [1] (see also [2]) and in our days is well known as Kapcinsky-Vladimirsky (KV) distribution.

Microcanonical distribution is widely applied in accelerator design as powerful tool for simulating of a charged particle beam of high intensity when particle interaction cannot be neglected.

Common desription of the microcanonical distribution is not quite correct from mathematical point of view because the mathematical expression for its phase density contains delta-function [2], which support is a point, while the support of the microcanonical distribution is three-dimensional surface of an ellipsoid in the four-dimensional phase space. In particular, such description does not allow to set initial distributions of particles for computer simulation of the beam.

Here we apply the covariant approach developed in the works [3, 4, 5, 6] for description of the microcanonical distribution. Covariant approach give us an opportunity to describe the distributions and to follow their evolution using various systems of coordinates in the phase space.

According to this approach, particle density is described by a differential form which degree is equal to dimension of support of the distribution.

If particles distributed in some region in the phase space, then dimension of their support is equal to dimension of the phase space and density of such distribution is described by a differential form of top degree, containing product of differentials of all coordinates. If support of a distribution is a surface in the phase space, then degree of the density form is less than dimension of the phase space, and such form can be written as a term containing only product of differentials of coordinates introduced on this surface. Most degenerate case is the case when particles are situated in some points of the phase space. In this case we should use density of a point-like measure which is specified by a scalar function, i.e., differential form of zero degree.

From this point of view, it is clear that microcanonical distribution should be described by a density specified on 3-dimensional surface of the ellipsoid, and this density is a form of 3 degree.

INTEGRALS OF PARTICLE MOTION IN LINEAR TRANSVERSE FIELD

Consider stationary charged particle beam in transverse electric field which is linear in transverse cartesian coordinates x and y

$$E_x = k_x x, \quad E_y = k_y y. \tag{1}$$

Assume that particles of the beam move with the same longitudinal velocity, and that at initial cross-section of the beam particles fill a four-dimensional ellipsoid in phase space of transverse motion

$$X^T B_0 X = 1, \quad X^* = (x, \dot{x}, y, \dot{y}).$$

Consider most common case $B_0 = \|b^0_{ik}\|$, $b^0_{ik} = b^0_{ki}$. If $b^0_{xy} \neq 0$, one can take other cartesian coordinates related with the old ones with orthogonal transformation in which $b^0_{xy} = 0$ (new coordinates are also regarded here as x, and y. Let introduce an assumption that $b^0_{x\dot{y}} = 0$, $b^0_{\dot{x},y} = 0$, $b^0_{\dot{x},\dot{y}} = 0$. Therefore, further we will consider the matrix of the form

$$B_0 = \begin{pmatrix} b^0_{xx} & b^0_{x\dot{x}} & 0 & 0 \\ b^0_{\dot{x}x} & b^0_{\dot{x}\dot{x}} & 0 & 0 \\ 0 & 0 & b^0_{yy} & b^0_{y\dot{y}} \\ 0 & 0 & b^0_{\dot{y}y} & b^0_{\dot{y}\dot{y}} \end{pmatrix}. \tag{2}$$

Assume also that equations of the transverse motion are linear:

$$\frac{d^2x}{dt^2} = -Q_x x, \quad \frac{d^2y}{dt^2} = -Q_y y. \tag{3}$$

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ISBN 978-3-95450-197-7

Then at each subsequent instant t particles fill ellipsoid

$$X^T(t)B(t)X(t) = 1,$$

$$B(t) = (F^T)^{-1}(t,t_0)B_0F^{-1}(t,t_0) = 1, \qquad (4)$$

where F is matrix Green function of the system of linear equations of the transverse motion (3). It is easy to see that if equation of the particle motion have form (3), matrix $B(t)$ has also form (2).

Assume that particles fill uniformly interior of the ellipse which is projection of the ellipsoid onto plane $\dot{x} = 0, \dot{y} = 0$:

$$\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} = 1,$$

where $R_{x,y}$ are semi-axes of the ellipse. Furter we shall see that this assumtion is confirmed also. Calculating the force acting on a particle from uniformly charged cylinder of elliptical cross-section we will see that

$$Q_x = \frac{e}{m}k_x + \frac{\lambda}{R_x(R_x+R_y)},$$

$$Q_y = \frac{e}{m}k_y + \frac{\lambda}{R_y(R_x+R_y)},$$

where λ is a coefficient depending on the beam current. Therefore, assumption that transverse force is linear is confirmed, if $B(t)$ changes sufficiently slowly.

To find beam envelopes, consider optimization problem

$$x(t) \to \max, \quad y(t) \to \max, \quad X^T(t)B(t)X(t) = 1,$$

Solving this problem, one can find that

$$R_x \equiv x_{\max}(t) = \sqrt{B^{-1}_{xx}(t)}, \qquad (5)$$

$$R_y \equiv y_{\max}(t) = \sqrt{B^{-1}_{yy}(t)}. \qquad (6)$$

Differentiating right sides of the expressions (5), (6), and considering motion along y-axis analogously, one can find that the envelopes R_x and $R_y \equiv y_{\max}(t)$ satisfy the equations

$$\frac{d^2R_x}{dt^2} = -Q_xR_x + \frac{S_x^2}{R_x^3}, \qquad (7)$$

$$\frac{d^2R_y}{dt^2} = -Q_yR_y + \frac{S_y^2}{R_y^3}, \qquad (8)$$

where $S_x = R_{x0}V_{x0}$, $S_y = R_{y0}V_{y0}$. Pairs of systems of equations for x, R_x, and for y, R_y form the Ermakov systems [7]. Their integrals, known also as Courant-Snyder invariants [8], are

$$I_x^2 = (x\dot{R}_x - \dot{x}R_x)^2 + \frac{S_x^2}{R_x^2}x^2, \qquad (9)$$

$$I_y^2 = (y\dot{R}_y - \dot{y}R_y)^2 + \frac{S_y^2}{R_y^2}y^2. \qquad (10)$$

COVARIANT DESCRIPTION OF MICROCANONICAL DISTRIBUTION

Let introduce coordinates $\varphi_x, \varphi_y, \theta$ on surface of the ellipsoid: (4)

$$x = R_x \cos\varphi_x \cos\theta, \qquad (11)$$

$$v_x \equiv x\dot{R}_x - \dot{x}R_x = E_x \sin\varphi_x \cos\theta, \qquad (12)$$

$$y = R_y \cos\varphi_y \sin\theta, \qquad (13)$$

$$v_y \equiv= y\dot{R}_y - \dot{y}R_y = E_y \sin\varphi_y \sin\theta. \qquad (14)$$

Substituting expressions (11)–(14) into (9),(10) one can see that θ is integral of motion.

Here we applied the covariant approach developed in works [3, 4, 5, 6], according to which distribution densities in the phase space are described by the differential form of various degrees satisfying to the Vlasov equation.

Let particles are distributed on surface of the ellipsoid (4) with the density

$$n_{\varphi_x\varphi_y\theta}\, d\varphi_x \wedge d\varphi_y \wedge d\theta. \qquad (15)$$

Let us find such $n_{\varphi_x\varphi_y\theta}$ that distribition of particles in projection of the ellipsoid to the plane x, y is uniform:

$$n_{xy}(x,y) = \begin{cases} \text{const}, & x^2/R_x^2 + y^2/R_y^2 \le 1, \\ 0, & x^2/R_x^2 + y^2/R_y^2 > 1. \end{cases}$$

From the other hand, one can take x, y, v_x as coordinates on surface of the ellipsoid. Density component in these coordinates can be obtained from relation

$$n_{\varphi_x\varphi_y\theta} = \varrho_{xyv_x} \cdot \left| \det \begin{pmatrix} \frac{\partial x}{\partial\varphi_x} & \frac{\partial y}{\partial\varphi_x} & \frac{\partial v_x}{\partial\varphi_x} \\ \frac{\partial x}{\partial\varphi_y} & \frac{\partial y}{\partial\varphi_y} & \frac{\partial v_x}{\partial\varphi_y} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} & \frac{\partial v_x}{\partial\theta} \end{pmatrix} \right| =$$

$$= n_{xyv_x} \cdot R_xR_yE_x \cdot |\sin\varphi_y| \cdot |\cos\theta| \sin^2\theta.$$

Expressing $\sin\varphi_y$ through these coordinates we have

$$|\sin\varphi_y| = \sqrt{1 - \frac{y^2}{R_y^2\sin^2\theta}} = \frac{\sqrt{R_y^2\sin^2\theta - y^2}}{R_y|\sin\theta|}.$$

Then

$$n_{xy}(x,y) = \int\limits_{-v_{x\max}(x,y)}^{v_{x\max}(x,y)} \varrho_{xyv_x}\, dv_x =$$

$$= \int\limits_{-v_{x\max}(x,y)}^{v_{x\max}(x,y)} \frac{n_{\varphi_x\varphi_y\theta}\, dv_x}{R_xE_x\sqrt{R_y^2\sin^2\theta - y^2} \cdot |\sin\theta\cos\theta|}.$$

Let

$$n_{\varphi_x\varphi_y\theta} = n_0|\sin\theta\cos\theta|. \qquad (16)$$

ISBN 978-3-95450-197-7

Then

$$n_{xy}(x,y) = \int\limits_{-v_{x\,\max}(x,y)}^{v_{x\,\max}(x,y)} \frac{n_0\, dv_x}{R_x E_x R_y} \times$$

$$\times \frac{1}{\sqrt{1 - (\frac{x}{R_x})^2 - (\frac{v_x}{E_x})^2 - (\frac{y}{R_y})^2}} = \frac{\pi n_0}{R_x R_y}.$$

Therefore the density component $\varrho_{\varphi_x \varphi_y \theta} = n_0 |\sin\theta \cos\theta|$ specifies uniform in the beam cross-section distribution. When particles move along beam axis, $n_{\varphi_x \varphi_y \theta}$ conserves as it depends only on motion integral θ.

To prove this strictly, substitute the density (16) into the Vlasov equation, which can be applied for degenerate distributions when it is written in the form [3, 4, 5, 6]

$$n(t + \delta t, F_{f\,\delta t} q) = F_{f,\,\delta t} n(t, q). \tag{17}$$

Here $F_{f\,\delta t}$ denotes the Lie dragging of a point of the phase space or some tensor object along vector field f by the parameter increment δt [9]. The vector field f here is defined by the dynamics equations (3).

In some coordinates specified on the surface where particle moves, equation (17) takes the form

$$\frac{\partial n}{\partial t} = -\mathcal{L}_f n(t, q). \tag{18}$$

Here $\mathcal{L}_f$ denotes the Lie derivative along vector field f [9].

In coordinates φ_x, φ_y, θ equation (18) reduces to the equation

$$\frac{\partial n}{\partial t} = \dot{\theta} \frac{\partial n}{\partial \theta},$$

and is satisfied, obviously. Therefore, assumption made before that particles are uniformely distributed inside ellipse

$$\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2} = 1$$

is confirmed.

CONCLUSION

So we can see that covariant approach allow us to describe correctly microcanonical distributions. We can find how the particle density on the surface of the ellipsoid depends on coordinates. Also we can describe evolution of this density in time. As we have obtained, this density is depends only on such function of coordinates which is integral of motion and, therefore, conserves. From practical point of view such approach give us an opportunity to simulate the microcanonical distribution setting initial distribution on the surface of the ellipsoid.

The results can be useful in many applied problems, for example, problems of optimization of beam transport channel.

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ISBN 978-3-95450-197-7