SYMMETRICAL PARAMETERIZATION FOR 6D FULLY COUPLED ONE-TURN TRANSPORT MATRIX

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ABSTRACT

Symmetry properties of 6D and 4D one-turn symplectic transport matrices were studied. A new parameterization was proposed for 6D matrix, which is an extension of the Lebedev—Bogacz parameterization for 4D case. The parameterization is fully symmetric relative to radial, vertical and longitudinal motion. It can be useful for lattices with strong coupling between all degrees of freedom.

BASIC DEFINITIONS

The only condition is that transport matrix $\mathbf{M}$ is symplectic. Therefore, its eigenvalues form 3 reciprocal pairs. The idea originally suggested by Mais and Ripken is to add $\mathbf{M}$ with its inverse:

$$\mathbf{M}^T\mathbf{S}_6\mathbf{M} = \mathbf{S}_6, \quad \mathbf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{S}_6 = \begin{pmatrix} \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S} \end{pmatrix}$$

$$\begin{cases} \mathbf{M}\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \\ \mathbf{M}\mathbf{v}_2 = \lambda_2\mathbf{v}_2 \\ \lambda_1\lambda_2 = 1 \end{cases} \Rightarrow \begin{cases} (\mathbf{M}+\mathbf{M}^{-1})\mathbf{v}_1 = (\lambda_1+\lambda_2)\mathbf{v}_1 \\ (\mathbf{M}+\mathbf{M}^{-1})\mathbf{v}_2 = (\lambda_1+\lambda_2)\mathbf{v}_2 \end{cases} \Rightarrow \begin{cases} \overleftrightarrow{\mathbf{M}}\mathbf{v}_1 = \hat\lambda\mathbf{v}_1 \\ \overleftrightarrow{\mathbf{M}}\mathbf{v}_2 = \hat\lambda\mathbf{v}_2 \end{cases}$$

Here $\overleftrightarrow{\mathbf{M}}$ is *recurrent matrix*

Transport matrix, its inverse and recurrent matrix can be written in a blockwise form:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} \end{pmatrix}, \quad \mathbf{M}^{-1} = \begin{pmatrix} \hat{\mathbf{M}}_{11} & \hat{\mathbf{M}}_{21} & \hat{\mathbf{M}}_{31} \\ \hat{\mathbf{M}}_{12} & \hat{\mathbf{M}}_{22} & \hat{\mathbf{M}}_{32} \\ \hat{\mathbf{M}}_{13} & \hat{\mathbf{M}}_{23} & \hat{\mathbf{M}}_{33} \end{pmatrix}, \quad \overleftrightarrow{\mathbf{M}} = \begin{pmatrix} b_1\mathbf{I} & \mathbf{R}_3 & \hat{\mathbf{R}}_2 \\ \hat{\mathbf{R}}_3 & b_2\mathbf{I} & \mathbf{R}_1 \\ \mathbf{R}_2 & \hat{\mathbf{R}}_1 & b_3\mathbf{I} \end{pmatrix},$$

where $\mathbf{R}_1 = \mathbf{M}_{23} + \hat{\mathbf{M}}_{32}$, $\mathbf{R}_2 = \mathbf{M}_{31} + \hat{\mathbf{M}}_{13}$, $\mathbf{R}_3 = \mathbf{M}_{12} + \hat{\mathbf{M}}_{21}$, $b_i = \mathrm{Tr}\,\mathbf{M}_{ii}$

and «^» is *pseudoinversion*: for any 2×2 matrix $\hat{\mathbf{A}} = -\mathbf{S}\mathbf{A}^T\mathbf{S}$.

therefore, $\mathbf{A} + \hat{\mathbf{A}} = (\mathrm{Tr}\,\mathbf{A})\mathbf{I}$, $\mathbf{A}\hat{\mathbf{A}} = |\mathbf{A}|\mathbf{I}$.

EIGENVALUES

Each eigenvector of $\overleftrightarrow{\mathbf{M}}$ can be splitted into 3 two-component subvectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$:

$$\overleftrightarrow{\mathbf{M}}\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} = \hat\lambda\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} \Rightarrow \begin{cases} (b_1-\hat\lambda)\mathbf{X} + \mathbf{R}_3\mathbf{Y} + \hat{\mathbf{R}}_2\mathbf{Z} = \bar{\mathbf{0}} \\ \hat{\mathbf{R}}_3\mathbf{X} + (b_2-\hat\lambda)\mathbf{Y} + \mathbf{R}_1\mathbf{Z} = \bar{\mathbf{0}} \\ \mathbf{R}_2\mathbf{X} + \hat{\mathbf{R}}_1\mathbf{Y} + (b_3-\hat\lambda)\mathbf{Z} = \bar{\mathbf{0}} \end{cases} \quad (1)$$

Each eigenvalue of $\overleftrightarrow{\mathbf{M}}$ is at least twice degenerated, therefore, $|\overleftrightarrow{\mathbf{M}} - \hat\lambda\mathbf{I}|$ is a perfect square of some polynomial with real coefficients. Then characteristic equation is:

$$\hat P(\hat\lambda) = |\mathbf{R}_1|(b_1-\hat\lambda) + |\mathbf{R}_2|(b_2-\hat\lambda) + |\mathbf{R}_3|(b_3-\hat\lambda) - (b_1-\hat\lambda)(b_2-\hat\lambda)(b_3-\hat\lambda) - \mathrm{Tr}(\mathbf{R}_1\mathbf{R}_2\mathbf{R}_3) = 0$$

3 roots $\hat\lambda_j$ can be found using Cardano formula. Motion is stable if and only if all of them are real and lie in the region (-2; 2). In non-degenerated case all $\hat\lambda_j$ are different, therefore all p_j are non-zero:

$$p_1 = \hat P'(\hat\lambda_1) = (\hat\lambda_2 - \hat\lambda_1)(\hat\lambda_3 - \hat\lambda_1), \quad p_2 = \hat P'(\hat\lambda_2) = (\hat\lambda_3 - \hat\lambda_2)(\hat\lambda_1 - \hat\lambda_2), \quad p_3 = \hat P'(\hat\lambda_3) = (\hat\lambda_1 - \hat\lambda_3)(\hat\lambda_2 - \hat\lambda_3)$$

EIGENVECTORS

System (1) can be rewritten as follows:

$$\begin{cases} u_{1j}\mathbf{Y} = \hat{\mathbf{W}}_{3j}\mathbf{X} \\ u_{1j}\mathbf{Z} = \mathbf{W}_{2j}\mathbf{X} \end{cases}, \text{ or } \begin{cases} u_{2j}\mathbf{Z} = \hat{\mathbf{W}}_{1j}\mathbf{Y} \\ u_{2j}\mathbf{X} = \mathbf{W}_{3j}\mathbf{Y} \end{cases}, \text{ or } \begin{cases} u_{3j}\mathbf{X} = \hat{\mathbf{W}}_{2j}\mathbf{Z} \\ u_{3j}\mathbf{Y} = \mathbf{W}_{1j}\mathbf{Z} \end{cases}, \text{ where}$$

$$\mathbf{W}_{1j} = \frac{1}{p_j}\left(\hat{\mathbf{R}}_3\hat{\mathbf{R}}_2 - (b_1-\hat\lambda_j)\mathbf{R}_1\right), \quad \mathbf{W}_{2j} = \frac{1}{p_j}\left(\hat{\mathbf{R}}_1\hat{\mathbf{R}}_3 - (b_2-\hat\lambda_j)\mathbf{R}_2\right), \quad \mathbf{W}_{3j} = \frac{1}{p_j}\left(\hat{\mathbf{R}}_2\hat{\mathbf{R}}_1 - (b_3-\hat\lambda_j)\mathbf{R}_3\right)$$

$$u_{1j} = \frac{1}{p_j}\left((b_2-\hat\lambda_j)(b_3-\hat\lambda_j) - |\mathbf{R}_1|\right), \quad u_{2j} = \frac{1}{p_j}\left((b_3-\hat\lambda_j)(b_1-\hat\lambda_j) - |\mathbf{R}_2|\right), \quad u_{3j} = \frac{1}{p_j}\left((b_1-\hat\lambda_j)(b_2-\hat\lambda_j) - |\mathbf{R}_3|\right)$$

So, 3 matrices can be formed, whose columns are eigenvectors of $\overleftrightarrow{\mathbf{M}}$:

$$\overleftrightarrow{\mathbf{W}}_1 = \begin{pmatrix} u_{11}\mathbf{I} & \mathbf{W}_{32} & \hat{\mathbf{W}}_{23} \\ \hat{\mathbf{W}}_{31} & u_{22}\mathbf{I} & \mathbf{W}_{13} \\ \mathbf{W}_{21} & \hat{\mathbf{W}}_{12} & u_{33}\mathbf{I} \end{pmatrix}, \quad \overleftrightarrow{\mathbf{W}}_2 = \begin{pmatrix} u_{12}\mathbf{I} & \mathbf{W}_{33} & \hat{\mathbf{W}}_{21} \\ \hat{\mathbf{W}}_{32} & u_{23}\mathbf{I} & \mathbf{W}_{11} \\ \mathbf{W}_{22} & \hat{\mathbf{W}}_{13} & u_{31}\mathbf{I} \end{pmatrix}, \quad \overleftrightarrow{\mathbf{W}}_1 = \begin{pmatrix} u_{13}\mathbf{I} & \mathbf{W}_{31} & \hat{\mathbf{W}}_{22} \\ \hat{\mathbf{W}}_{33} & u_{21}\mathbf{I} & \mathbf{W}_{12} \\ \mathbf{W}_{23} & \hat{\mathbf{W}}_{11} & u_{32}\mathbf{I} \end{pmatrix}$$

u_{ij} are *coupling coefficients.* $\mathbf{W}_{ij}$ and u_{ij} have the following properties:

$$\mathbf{W}_{2j}\mathbf{W}_{3j} = u_{1j}\hat{\mathbf{W}}_{1j}, \quad \mathbf{W}_{3j}\mathbf{W}_{1j} = u_{2j}\hat{\mathbf{W}}_{2j}, \quad \mathbf{W}_{1j}\mathbf{W}_{2j} = u_{3j}\hat{\mathbf{W}}_{3j}$$

$$|\mathbf{W}_{1j}| = u_{2j}u_{3j}, \quad |\mathbf{W}_{2j}| = u_{3j}u_{1j}, \quad |\mathbf{W}_{3j}| = u_{1j}u_{2j}, \quad \mathbf{W}_{i1} + \mathbf{W}_{i2} + \mathbf{W}_{i3} = \mathbf{0} \quad \Rightarrow \quad \boxed{\overleftrightarrow{\mathbf{W}}_1 + \overleftrightarrow{\mathbf{W}}_2 + \overleftrightarrow{\mathbf{W}}_3 = \mathbf{I}_6}$$

$u_{i1} + u_{i2} + u_{i3} = u_{1j} + u_{2j} + u_{3j} = 1 \Rightarrow u_{32} - u_{23} = u_{13} - u_{31} = u_{21} - u_{12} = l$, *coupling asymmetry.*

So, all u_{ij} can be expressed in terms of u_{11}, u_{22}, u_{33} and l.

TWISS PARAMETERIZATION

Matrices $\overleftrightarrow{\mathbf{W}}_j$ reduce transport matrix to the block-diagonal form:

$$\mathbf{T}_1 = \overleftrightarrow{\mathbf{W}}_1^{-1}\mathbf{M}\overleftrightarrow{\mathbf{W}}_1 = \begin{pmatrix} \mathbf{T}_{11} & 0 & 0 \\ 0 & \mathbf{T}_{22} & 0 \\ 0 & 0 & \mathbf{T}_{33} \end{pmatrix}, \quad \mathbf{T}_2 = \overleftrightarrow{\mathbf{W}}_2^{-1}\mathbf{M}\overleftrightarrow{\mathbf{W}}_2 = \begin{pmatrix} \mathbf{T}_{12} & 0 & 0 \\ 0 & \mathbf{T}_{23} & 0 \\ 0 & 0 & \mathbf{T}_{31} \end{pmatrix}, \quad \mathbf{T}_3 = \overleftrightarrow{\mathbf{W}}_3^{-1}\mathbf{M}\overleftrightarrow{\mathbf{W}}_3 = \begin{pmatrix} \mathbf{T}_{13} & 0 & 0 \\ 0 & \mathbf{T}_{21} & 0 \\ 0 & 0 & \mathbf{T}_{32} \end{pmatrix}$$

Well-known Twiss parameterization can be introduced for each diagonal block:

$$\mathbf{T}_{ij} = \mathbf{I}\cos\mu_j + \mathbf{J}_{ij}\sin\mu_j, \quad \mathbf{J}_{ij} = \begin{pmatrix} \alpha_{ij} & \beta_{ij} \\ -\gamma_{ij} & -\alpha_{ij} \end{pmatrix}, \quad \gamma_{ij} = \frac{1+\alpha_{ij}^2}{\beta_{ij}}, \quad \mu_j = \arg\lambda_j$$

There are commutation rules for J- and W-matrices:

$$\mathbf{W}_{1j}\mathbf{J}_{3j} = \mathbf{J}_{2j}\mathbf{W}_{1j}, \quad \mathbf{W}_{2j}\mathbf{J}_{1j} = \mathbf{J}_{3j}\mathbf{W}_{2j}, \quad \mathbf{W}_{3j}\mathbf{J}_{2j} = \mathbf{J}_{1j}\mathbf{W}_{3j}$$

Now closed expression for transport matrix can be found: $\boxed{\mathbf{M} = \overleftrightarrow{\mathbf{W}}_1\mathbf{T}_1 + \overleftrightarrow{\mathbf{W}}_2\mathbf{T}_2 + \overleftrightarrow{\mathbf{W}}_3\mathbf{T}_3}$

W-MATRICES, 6D CASE

All W-matrices can be expressed in terms of $\mathbf{W}_{11}, \mathbf{W}_{22}, \mathbf{W}_{33}$:

$$\begin{cases} l\mathbf{W}_{13} = u_{23}\mathbf{W}_{11} - \hat{\mathbf{W}}_{33}\hat{\mathbf{W}}_{22} \\ l\mathbf{W}_{12} = -u_{32}\mathbf{W}_{11} + \hat{\mathbf{W}}_{33}\hat{\mathbf{W}}_{22} \\ l\mathbf{W}_{21} = u_{31}\mathbf{W}_{22} - \hat{\mathbf{W}}_{11}\hat{\mathbf{W}}_{33} \\ l\mathbf{W}_{23} = -u_{13}\mathbf{W}_{22} + \hat{\mathbf{W}}_{11}\hat{\mathbf{W}}_{33} \\ l\mathbf{W}_{32} = u_{12}\mathbf{W}_{33} - \hat{\mathbf{W}}_{22}\hat{\mathbf{W}}_{11} \\ l\mathbf{W}_{31} = -u_{21}\mathbf{W}_{33} + \hat{\mathbf{W}}_{22}\hat{\mathbf{W}}_{11} \end{cases} \text{ or } \begin{cases} \mathrm{Tr}(\mathbf{J}_{22}\mathbf{J}_{23} - \mathbf{J}_{32}\mathbf{J}_{33})\mathbf{W}_{13} = \mathbf{J}_{23}(\mathbf{W}_{11}\mathbf{J}_{32} - \mathbf{J}_{22}\mathbf{W}_{11}) + (\mathbf{W}_{11}\mathbf{J}_{32} - \mathbf{J}_{22}\mathbf{W}_{11})\mathbf{J}_{33} \\ \mathrm{Tr}(\mathbf{J}_{22}\mathbf{J}_{23} - \mathbf{J}_{32}\mathbf{J}_{33})\mathbf{W}_{12} = \mathbf{J}_{22}(\mathbf{W}_{11}\mathbf{J}_{33} - \mathbf{J}_{23}\mathbf{W}_{11}) + (\mathbf{W}_{11}\mathbf{J}_{33} - \mathbf{J}_{23}\mathbf{W}_{11})\mathbf{J}_{32} \\ \mathrm{Tr}(\mathbf{J}_{33}\mathbf{J}_{31} - \mathbf{J}_{13}\mathbf{J}_{11})\mathbf{W}_{21} = \mathbf{J}_{31}(\mathbf{W}_{22}\mathbf{J}_{13} - \mathbf{J}_{33}\mathbf{W}_{22}) + (\mathbf{W}_{22}\mathbf{J}_{13} - \mathbf{J}_{33}\mathbf{W}_{22})\mathbf{J}_{11} \\ \mathrm{Tr}(\mathbf{J}_{33}\mathbf{J}_{31} - \mathbf{J}_{13}\mathbf{J}_{11})\mathbf{W}_{23} = \mathbf{J}_{33}(\mathbf{W}_{22}\mathbf{J}_{11} - \mathbf{J}_{31}\mathbf{W}_{22}) + (\mathbf{W}_{22}\mathbf{J}_{11} - \mathbf{J}_{31}\mathbf{W}_{22})\mathbf{J}_{13} \\ \mathrm{Tr}(\mathbf{J}_{11}\mathbf{J}_{12} - \mathbf{J}_{21}\mathbf{J}_{22})\mathbf{W}_{32} = \mathbf{J}_{12}(\mathbf{W}_{33}\mathbf{J}_{21} - \mathbf{J}_{11}\mathbf{W}_{33}) + (\mathbf{W}_{33}\mathbf{J}_{21} - \mathbf{J}_{11}\mathbf{W}_{33})\mathbf{J}_{22} \\ \mathrm{Tr}(\mathbf{J}_{11}\mathbf{J}_{12} - \mathbf{J}_{21}\mathbf{J}_{22})\mathbf{W}_{31} = \mathbf{J}_{11}(\mathbf{W}_{33}\mathbf{J}_{22} - \mathbf{J}_{12}\mathbf{W}_{33}) + (\mathbf{W}_{33}\mathbf{J}_{22} - \mathbf{J}_{12}\mathbf{W}_{33})\mathbf{J}_{21} \end{cases}$$

Each $\mathbf{W}_{jj}$ can be parameterized with 1 parameter φ_j:

$$\begin{cases} \mathbf{W}_{11} = r_{11}(\mathbf{I}\cos\phi_1 + \mathbf{J}_{21}\sin\phi_1)(\mathbf{J}_{21} + \mathbf{J}_{31}) \\ \mathbf{W}_{22} = r_{22}(\mathbf{I}\cos\phi_2 + \mathbf{J}_{32}\sin\phi_2)(\mathbf{J}_{32} + \mathbf{J}_{12}), \\ \mathbf{W}_{33} = r_{33}(\mathbf{I}\cos\phi_3 + \mathbf{J}_{13}\sin\phi_3)(\mathbf{J}_{13} + \mathbf{J}_{23}) \end{cases} \quad r_{11} = \sqrt{\frac{u_{21}u_{31}}{2 - \mathrm{Tr}(\mathbf{J}_{21}\mathbf{J}_{31})}} \quad r_{22} = \sqrt{\frac{u_{32}u_{12}}{2 - \mathrm{Tr}(\mathbf{J}_{32}\mathbf{J}_{12})}} \quad r_{33} = \sqrt{\frac{u_{13}u_{23}}{2 - \mathrm{Tr}(\mathbf{J}_{13}\mathbf{J}_{23})}}$$

Some additional notations are needed to express φ_j (formulae only for φ_3 are presented here, for φ_1 and φ_2 cyclic permutations of indices should be applied):

$$\mathbf{A}_3' = r_{33}\left((\mathbf{J}_{13} + \mathbf{J}_{23})\mathbf{J}_{21} - \mathbf{J}_{11}(\mathbf{J}_{13} + \mathbf{J}_{23})\right), \quad \mathbf{A}_3'' = r_{33}\left((\mathbf{J}_{13}\mathbf{J}_{23} - \mathbf{I})\mathbf{J}_{21} - \mathbf{J}_{11}(\mathbf{J}_{13}\mathbf{J}_{23} - \mathbf{I})\right)$$

$$\mathbf{B}_3' = r_{33}\left((\mathbf{J}_{13} + \mathbf{J}_{23})\mathbf{J}_{22} - \mathbf{J}_{12}(\mathbf{J}_{13} + \mathbf{J}_{23})\right), \quad \mathbf{B}_3'' = r_{33}\left((\mathbf{J}_{13}\mathbf{J}_{23} - \mathbf{I})\mathbf{J}_{22} - \mathbf{J}_{12}(\mathbf{J}_{13}\mathbf{J}_{23} - \mathbf{I})\right)$$

$$\mathbf{C}_3' = \mathbf{J}_{12}\mathbf{A}_3' + \mathbf{A}_3'\mathbf{J}_{22}, \quad \mathbf{C}_3'' = \mathbf{J}_{12}\mathbf{A}_3'' + \mathbf{A}_3''\mathbf{J}_{22}, \quad q_3 = u_{12}u_{22}\left(\mathrm{Tr}(\mathbf{J}_{11}\mathbf{J}_{12} - \mathbf{J}_{21}\mathbf{J}_{22})\right)^2$$

$$\mathbf{D}_3' = \mathbf{J}_{11}\mathbf{B}_3' + \mathbf{B}_3'\mathbf{J}_{21}, \quad \mathbf{D}_3'' = \mathbf{J}_{11}\mathbf{B}_3'' + \mathbf{B}_3''\mathbf{J}_{21}, \quad q_3' = u_{11}u_{21}\left(\mathrm{Tr}(\mathbf{J}_{11}\mathbf{J}_{12} - \mathbf{J}_{21}\mathbf{J}_{22})\right)^2$$

$$\begin{matrix} f_3 = |\mathbf{C}_3''| - q_3, & g_3 = |\mathbf{C}_3'| - q_3, & h_3 = \mathrm{Tr}(\mathbf{C}_3'\hat{\mathbf{C}}_3'') \\ f_3' = |\mathbf{D}_3''| - q_3', & g_3' = |\mathbf{D}_3'| - q_3', & h_3' = \mathrm{Tr}(\mathbf{D}_3'\hat{\mathbf{D}}_3''), \end{matrix} \quad \phi_3 = \frac{f_3g_3' - f_3'g_3}{h_3f_3' - h_3'f_3} + \left(\frac{1}{2} \pm \frac{1}{2}\right)\pi$$

There are 3 ways of treating these «±»:

1. Include φ_1, φ_2, φ_3 into the parameterization as dependent parameters.
2. Introduce 3 boolean parameters to indicate signs of φ_j.
3. For each «+» change signs of corresponding μ_j, α_{ij}, β_{ij} and then change «+» to «-».

Finally, the parameterization has 25 parameters: 3 μ_j, 9 α_{ij}, 9 β_{ij}, 3 u_{ii} and l. 21 of them are independent, and there are 4 identities:

$$\mathrm{Tr}(\mathbf{W}_{11}\mathbf{J}_{33}\hat{\mathbf{W}}_{11}\mathbf{J}_{22} - \mathbf{W}_{11}\mathbf{J}_{32}\hat{\mathbf{W}}_{11}\mathbf{J}_{23}) = (u_{23}u_{33} - u_{22}u_{32})\mathrm{Tr}(\mathbf{J}_{22}\mathbf{J}_{23} - \mathbf{J}_{32}\mathbf{J}_{33})$$

$$\mathrm{Tr}(\mathbf{W}_{22}\mathbf{J}_{11}\hat{\mathbf{W}}_{22}\mathbf{J}_{33} - \mathbf{W}_{22}\mathbf{J}_{13}\hat{\mathbf{W}}_{22}\mathbf{J}_{31}) = (u_{31}u_{11} - u_{33}u_{13})\mathrm{Tr}(\mathbf{J}_{33}\mathbf{J}_{31} - \mathbf{J}_{13}\mathbf{J}_{11})$$

$$\mathrm{Tr}(\mathbf{W}_{33}\mathbf{J}_{22}\hat{\mathbf{W}}_{33}\mathbf{J}_{11} - \mathbf{W}_{33}\mathbf{J}_{21}\hat{\mathbf{W}}_{33}\mathbf{J}_{12}) = (u_{12}u_{22} - u_{11}u_{21})\mathrm{Tr}(\mathbf{J}_{11}\mathbf{J}_{12} - \mathbf{J}_{21}\mathbf{J}_{22})$$

$$4\,\mathrm{Tr}(\mathbf{W}_{11}\mathbf{W}_{22}\mathbf{W}_{33}) = (1 + u_{11} - u_{22} - u_{33})(1 + u_{22} - u_{33} - u_{11})(1 + u_{33} - u_{11} - u_{22}) - l^2(1 + u_{11} + u_{22} + u_{33})$$

W-MATRICES, 4D CASE

4D case can be deduced from 6D case. W-matrices have the following properties:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{0} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{33} \end{pmatrix} \Rightarrow \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} = \begin{pmatrix} 1-u & u & 0 \\ u & 1-u & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{W}\mathbf{J}_{21} = \mathbf{J}_{11}\mathbf{W}, \quad \mathbf{W}\mathbf{J}_{22} = \mathbf{J}_{12}\mathbf{W}, \quad |\mathbf{W}| = u(1-u), \quad \mathbf{W} = \mathbf{W}_{32} = -\mathbf{W}_{31}, \quad \text{other } \mathbf{W}_{ij} = \mathbf{0}$$

If $|\mathbf{W}| \neq 0$, then this system can be solved only in case of $\mathrm{Tr}(\mathbf{J}_{11}\mathbf{J}_{12} - \mathbf{J}_{21}\mathbf{J}_{22}) = 0$.

So, $\beta_{11}\gamma_{12} + \beta_{12}\gamma_{11} - 2\alpha_{11}\alpha_{12} = \beta_{21}\gamma_{22} + \beta_{22}\gamma_{21} - 2\alpha_{21}\alpha_{22}$ is an invariant.

And the solution is:

$$\mathbf{W} = \pm\sqrt{\frac{u(1-u)}{|\mathbf{J}_{11}(\mathbf{J}_{12} - \mathbf{J}_{22}) + (\mathbf{J}_{12} - \mathbf{J}_{22})\mathbf{J}_{21}|}}\left(\mathbf{J}_{11}(\mathbf{J}_{12} - \mathbf{J}_{22}) + (\mathbf{J}_{12} - \mathbf{J}_{22})\mathbf{J}_{21}\right)$$

Signs of α_{11}, α_{21}, β_{11}, β_{21}, μ_1 or α_{12}, α_{22}, β_{12}, β_{22}, μ_2, should be changed in case of «-», then «±» should be omitted.

LEBEDEV—BOGACZ PARAMETERIZATION

There is a correspondence between Lebedev–Bogacz's (left) and our notation (right):

$$\mu_{1L} = \mu_1\,\mathrm{sgn}\,\beta_{11} \quad \mu_{2L} = \mu_2\,\mathrm{sgn}\,\beta_{22}$$

$$\beta_{1x} = |(1-u)\,\mathrm{sgn}\,\beta_{11}| \quad \beta_{1y} = |u\,\mathrm{sgn}\,\beta_{21}|$$

$$\beta_{2y} = |(1-u)\,\mathrm{sgn}\,\beta_{22}| \quad \beta_{2x} = |u\,\mathrm{sgn}\,\beta_{12}|$$

$$\alpha_{1x} = (1-u)\alpha_{11}\,\mathrm{sgn}((1-u)\beta_{11}) \quad \alpha_{2x} = u\alpha_{12}\,\mathrm{sgn}(u\beta_{12})$$

$$\alpha_{2y} = (1-u)\alpha_{22}\,\mathrm{sgn}((1-u)\beta_{22}) \quad \alpha_{1y} = u\alpha_{21}\,\mathrm{sgn}(u\beta_{21})$$

Lebedev–Bogacz's notation:

- u depends on α_{ij} and β_{ij}
- $\beta_{ij} > 0$
- 2 additional boolean parameters

Our notation:

- u is independent parameter, but there is one invariant involving α_{ij} and β_{ij}
- β_{ij} may be negative
- No additional boolean parameters

SECOND MOMENTS, EMITTANCES

Second-moments matrix $\boldsymbol{\Sigma}$ meets the following condition: $\boldsymbol{\Sigma} = \mathbf{M}\boldsymbol{\Sigma}\mathbf{M}^T$

It can be expressed in terms of $\overleftrightarrow{\mathbf{W}}_j$:

$$\boxed{\boldsymbol{\Sigma} = \overleftrightarrow{\mathbf{W}}_1\tilde{\boldsymbol{\Sigma}}_1 + \overleftrightarrow{\mathbf{W}}_2\tilde{\boldsymbol{\Sigma}}_2 + \overleftrightarrow{\mathbf{W}}_3\tilde{\boldsymbol{\Sigma}}_3}$$

Here $\tilde{\boldsymbol{\Sigma}}_j$ are block-diagonal matrices:

$$\tilde{\boldsymbol{\Sigma}}_1 = \begin{pmatrix} \tilde{\boldsymbol{\Sigma}}_{11} & 0 & 0 \\ 0 & \tilde{\boldsymbol{\Sigma}}_{22} & 0 \\ 0 & 0 & \tilde{\boldsymbol{\Sigma}}_{33} \end{pmatrix}, \quad \tilde{\boldsymbol{\Sigma}}_2 = \begin{pmatrix} \tilde{\boldsymbol{\Sigma}}_{12} & 0 & 0 \\ 0 & \tilde{\boldsymbol{\Sigma}}_{23} & 0 \\ 0 & 0 & \tilde{\boldsymbol{\Sigma}}_{31} \end{pmatrix}, \quad \tilde{\boldsymbol{\Sigma}}_3 = \begin{pmatrix} \tilde{\boldsymbol{\Sigma}}_{13} & 0 & 0 \\ 0 & \tilde{\boldsymbol{\Sigma}}_{21} & 0 \\ 0 & 0 & \tilde{\boldsymbol{\Sigma}}_{32} \end{pmatrix}, \quad \tilde{\boldsymbol{\Sigma}}_{ij} = \varepsilon_j\begin{pmatrix} \beta_{ij} & -\alpha_{ij} \\ -\alpha_{ij} & \gamma_{ij} \end{pmatrix}$$

ε_j are emittances of normal modes, they can be calculated from beam sizes:

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} = \begin{pmatrix} u_{11}\beta_{11} & u_{12}\beta_{12} & u_{13}\beta_{13} \\ u_{21}\beta_{21} & u_{22}\beta_{22} & u_{23}\beta_{23} \\ u_{31}\beta_{31} & u_{32}\beta_{32} & u_{33}\beta_{33} \end{pmatrix}^{-1}\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$$