ON THE INTEGRO-DIFFERENTIAL EQUATIONS FOR DYNAMICS OF INTERACTING CHARGED PARTICLES MODELING

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Abstract

In this paper we consider some integral-differential model of the dynamics of charged particles with smoothed interaction. This model is used in solving various problems of optimization of the dynamics of intense beams. Using the proposed model in optimization problems allows you to find analytical expressions for the functional variation that characterize the dynamics of the particles, and then consruct methods of directed search of extremum.

INTRODUCTION

Problems of the analysis of charged particles dynamics in view of their interaction have long been the focus of many researchers. One of the basic mathematical models describing the dynamics of the interaction of particles is the mathematical model proposed by A.A.Vlasov [1]. Vlasov equation widely used to solve a variety of application problems. Of particular interest is the finding of the self-consistent distributions to a beam of charged particles in an electromagnetic field [2-4,15]. The problems of existence and uniqueness of solutions of the Vlasov equation considered in [5,6]. It should be noted that in the numerical simulation of the dynamics of intense beams mainly smoothed interaction of charged particles is used [7-10]. In this paper we consider some integral-differential model of the dynamics of charged particles with smoothed interaction. This model is used in solving various problems of optimization of the dynamics of intense beams. Using the proposed model in optimization problems allows you to find analytical expressions for the functional variation that characterize the dynamics of the particles, and then construct methods of directed search of extremum [11-14]. The paper describes an example of the construction of such integral-differential model for the dynamics of charged particles.

INTEGRO-DIFFERENTIAL MODEL

Suppose that the dynamics of the beam of interacting charged particles is described by the system of integro-differential equations

$$\frac{dx}{dt} = f(t,x), \tag{1}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x} f(t,x) + \rho\, div_x f(t,x) = 0, \tag{2}$$

$$f(t,x) = f_1(t,x) + \int_{M_t} \rho(t,y) f_2(t,x,y)\, dy \tag{3}$$

with initial conditions

$$x(t_0,x_0) = x_0 \in \overline{M}_0, \quad \rho(t_0,x) = \rho_0(x). \tag{4}$$

Here the nonempty open bounded set $M_0 \subset R^n$; the real-valued nonnegative continuous function $\rho_0(x)$ in $\overline{M}_0$ specifies a density of particle distribution in the phase space at the initial time t_0; the vector-function $f_1(t,x)$ is determined by the external electromagnetic fields acting on particles; the vector-function $f_2(t,x,y)$ is determined by considering the particle interaction. Solution of (1)-(4) represent a set of vector-functions $x(t,x_0)$ that determine the bundle of trajectories emanating from the set M_0. Note that $M_t = \{x(t,x_0) : x_0 \in M_0\}$) and $\rho(t,x(t,x_0))$ is the density of particle distribution along these trajectories. Equality (2) means that

$$\int_{M_t} \rho(t,y) f_2(t,x,y)\, dy = \int_{M_0} \rho_0(y_0) f_2(t,x,x(t,y_0))\, dy_0,$$

that is, we consider the system of integro-differential equations

$$\frac{dx(t,x_0)}{dt} = f(t,x(t,x_0)) = f_1(t,x(t,x_0)) + \int_{M_0} \rho_0(y_0) f_2(t,x(t,x_0),x(t,y_0)) dy_0 \tag{5}$$

with initial conditions

$$x(t_0,x_0) = x_0 \in \overline{M}_0. \tag{6}$$

Suppose that the vector real functions $f_1(t,x)$ and $f_2(t,x,y)$ are defined and continuous on the sets $(\alpha,\beta)\times\Omega$ and $(\alpha,\beta)\times\Omega\times\Omega$ respectively, where $(\alpha,\beta) \in R^1$, and Ω is a region in R^n.

Denote $R_a = \{t : |t-t_0| \le a\}$, $\overline{M}_b = \{x : ||x-x_0|| \le b,\ x_0 \in \overline{M}_0\}$, $R_1 = R_a \times \overline{M}_b$, $R_2 = R_a \times \overline{M}_b \times \overline{M}_b$.

We have the following theorem of existence and uniqueness.

Theorem Suppose the following conditions are satisfied:
1) the nonnegative function $\rho_0(x) \in C(\overline{M}_0)$ is given: $\rho_0(x) \ne 0$ for $x \in M_0$, and $\int_{M_0} \rho_0(x)dx = \rho < +\infty$;
2) numbers $a > 0$ and $b > 0$ are given, such that $R_a \subset (\alpha,\beta)$, $\overline{M}_b \subset \Omega$;
3) $M_1 = \sup_{(t,x)\in R_1} ||f_1(t,x)||$, $M_2 = \sup_{(t,x,y)\in R_2} ||f_2(t,x,y)||$;
4) the vector-functions $f_1((t,x)$ and $f_2((t,x,y)$ satisfy the Lipschitz condition in the variables x and x,y with constant L_1 and L_2 on the sets R_1 and R_2, respectively.

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Then there exists a unique vector-function $x(t, x_0)$ that is defined and continuously differentiable with respect to t; it continuous in x_0 on $R_h \times \overline{M}_0$ and satisfies equation (5) and initial conditions $x(t_0, x_0) = x_0$. Here $h = \min(a, b/M)$ and $M = M_1 + \rho M_2$.

The proof of this theorem can be found in [9].

Let us assume that the vector-functions $f_1((t, x)$ and $f_2((t, x, y)$ are continuously differentiable with respect to x and y. Then the solution $x(t, x_0)$ of system (5) is continuously differentiable with respect to x_0. This assertion is proved just as the continuous differentiability of solutions for the systems of ordinary differential equations is proved with respect to initial data. In this case the matrix $\partial x(t, x_0)/\partial x_0$ satisfies the equation

$$\frac{d}{dt}\frac{\partial x(t,x_0)}{\partial x_0} = \frac{\partial f(t,x(t,x_0))}{\partial x}\frac{\partial x(t,x_0)}{\partial x_0} = \left(\frac{\partial f_1(t,x(t,x_0))}{\partial x} + \int\limits_{M_t} \frac{\partial f_2(t,x(t,x_0),y_t)}{\partial x}\rho(t,y_t)\,dy_t\right)\frac{\partial x(t,x_0)}{\partial x_0}$$

The existence and uniqueness of the solution $\rho(t, x)$ of equation (2) follows [9] from the equality

$$\frac{d\rho(t,x(t,x_0))}{dt} = -\rho(t,x(t,x_0))\mathrm{div}_x f(t,x(t,x_0))$$

and continuous differentiability of $x(t, x_0)$ to the initial data.

Problem (1)-(3) becomes the problem of controlling an ensemble of trajectories [9], if the function $f_1(t, x)$ depends on the control u (as a rule, the parameters of the accelerator), that is $f_1 = f_1(t, x, u)$, $x = x(t, x_0, u)$, and $M_t = M_{t,u} = \{x_t = x(t, x_0, u) : x_0 \in \overline{M}_0\}$. The quality of the beam dynamics of charged particles may be evaluated by functional such as

$$I(u) = \int\limits_0^T \int\limits_{M_{t,u}} \phi(t,x_t,\rho(t,x_t))\,dx_t dt + \int\limits_{M_{T,u}} g(x_T,\rho(T,x_T))\,dx_T, \quad (7)$$

where $\phi(t, x, \rho)$ and $g(x, \rho)$ are nonnegative continuously differentiable in its arguments functions. The control u we choose minimizing the functional (7). In the case of continuous differentiable functions f_1, f_2, $\mathrm{div}_x f_1$, $\mathrm{div}_x f_2$ it is possible to obtain an analytical expression [9] for the variation of the functional and thus construct directed methods to minimize it.

Here is an example of a model of the form (5) with smooth interaction, that is, with continuously differentiable functions $f_2(t, x, y)$, $\mathrm{div}_x f_2$. This example shows that the methods of large particles can be formulated in integro-differential form.

INTEGRO-DIFFERENTIAL DISK INTERACTION MODEL

In studies of the longitudinal motion of charged particles in axially symmetric external electromagnetic fields the particle beam is often seen as a set of N disks of radius R. Each disk moves at a time t along the axis z accelerating structure under the action of electromagnetic field generated in the accelerator and under the action of the field created by the remaining disks. The equations of motion i-th disc in dimensionless coordinates are

$$\frac{d\xi_i}{d\tau} = \frac{p_i}{\sqrt{1+p_i^2}}, \quad (8)$$

$$\frac{dp_i}{d\tau} = \alpha(\tau,\xi_i) + F_i. \quad (9)$$

Here $p_i = \beta_i/\sqrt{1-\beta_i^2}$ is the i-th particle momentum; $\beta_i = v_i/c$; v_i is the velocity of the i-th disk along the axis ξ; $F_i = \sum_{J=1}^{N} F_{ij}$, where F_{ij} is the force with which the j-th disc acts on the I-th disk. In calculating the force of one disk on the other along the ξ axis, we assume that the motion of the disk is uniform and the potential of any disk circle is given [7] by equality

$$U(r,z) = \frac{R_0}{\epsilon_0 a}\sum_{i=1}^{\infty}\frac{J_0(\mu_i R_0/a)J_0(\mu_i r/a)}{\mu_i J_1^2(\mu_i)}e^{-\mu_i|z|/a}. \quad (10)$$

Here R_0 is the radius of a charged circle; a is the radius of the tube; J_0 and J_1 are the Bessel functions; μ_i are the roots of the function J_0; r is the distance of the observation point from the tube axis, z is the longitudinal coordinate of the point at which the potential is calculated.

Expression (10) is valid for each circle in its own system of coordinates. That is, we need to calculate $-\partial U/\partial\xi$, to integrate with respect to the disk thickness and radius, and then to pass to a stationary system of coordinates.

For thin disks, the force with which the j-th thin disk acts on the i-th thin disk

$$f(\xi_i - \xi_j) = \frac{2e^2}{\pi R^2\epsilon_0}\mathrm{sign}(\xi_i - \xi_j)\times \sum_{i=1}^{\infty}\left(\frac{J_1(\mu_i R/a)}{\mu_i J_1(\mu_i)}\right)^2 e^{-\mu_i\lambda|z_2 - z_1|/a}.$$

is the discontinuous function at $\xi_i = \xi_j$, but for disks of finite thickness $2d$ the function $F(\xi_i - \xi_j)$ is continuously differentiable [9] (smoothed interaction). Here ξ_i and ξ_j are the coordinates of the centers of the thick discs in the stationary coordinate system.

In the simulation of particle dynamics taking into account the relativistic effects $F = F(\xi_i - \xi_j, p_j)$.

Thus, F_i in the equation (9) for the phase coordinate p_i has the form

$$F_i = \sum_{j=1}^{N} F(\xi_i - \xi_j, p_j)$$

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Let points $\{(\xi_{i0}, p_{i0})\}_{i=1}^{N}$ are distributed in the set $\overline{M}_0$ with partial density $\rho_0(\xi, p)$, and let $(\xi_i(\tau, \xi_{i0}, p_{i0}), p_i(\tau, \xi_{i0}, p_{i0}))$ is the solution of a system (8)-(9) with the initial conditions

$$\xi_i(\tau_0, \xi_{i0}, p_{i0}) = \xi_{i0},\ \ p(\tau, \xi_{i0}, p_{i0}) = p_{i0},\ \ i = 1, 2, \dots, N.$$

When $N \to \infty$ the sum F_i should be replaced by the integral

$$\iint\limits_{M_\tau} \rho(\tau, \xi', p') F(\xi(\tau, \xi_0, p_0) - \xi', p') d\xi dp',$$

where $\xi(\tau, \xi_0, p_0)$, $p(\tau, \xi_0, p_0)$ satisfy the system of equations

$$\frac{d\xi(\tau, \xi_0, p_0)}{d\tau} = \frac{p(\tau, \xi_0, p_0)}{(1 + p^2(\tau, \xi_0, p_0))^{1/2}},$$

$$\frac{dp}{d\tau} = \alpha(\tau, \xi_i) + \iint\limits_{M_\tau} \rho(\tau, \xi', p') F(\xi(\tau, \xi_0, p_0) - \xi', p') d\xi' dp'.$$

Here

$$M_\tau = \{(\xi(\tau, \xi_0, p_0), p(\tau, \xi_0, p_0) : (\xi_0, p_0) \in \overline{M}_0\}.$$

Note that

$$\rho(\tau, \xi(\tau, \xi_0', p_0'), p(\tau, \xi_0', p_0')) = \rho_0(\xi_0', p_0') \times \det{}^{-1} \left(\frac{D(\xi(\tau, \xi_0', p_0'), p(\tau, \xi_0', p_0'))}{D(\xi_0', p_0')} \right)$$

and, therefore,

$$\iint\limits_{M_\tau} \rho(\tau, \xi', p') G(\xi(\tau, \xi_0, p_0) - \xi', p') d\xi' dp' = \iint\limits_{M_0} \rho_0(\xi_0', p_0') F(\xi(\tau, \xi_0, p_0) - \xi(\tau, \xi_0', p_0'), p(\tau, \xi_0', p_0')) d\xi_0' dp_0'.$$

Thus, we obtain the system of the form (5):

$$\frac{d\xi(\tau, \xi_0, p_0)}{d\tau} = \frac{p(\tau, \xi_0, p_0)}{(1 + p^2(\tau, \xi_0, p_0))^{1/2}},$$

$$\frac{dp(\tau, \xi_0, p_0)}{d\tau} = \alpha(\tau, \xi(\tau, \xi_0, p_0)) + \iint\limits_{M_0} \rho_0(\xi_0', p_0') F(\xi(\tau, \xi_0, p_0) - \xi(\tau, \xi_0', p_0'), p(\tau, \xi_0', p_0')) d\xi_0' dp_0'.$$

The one-dimensional disk model is convenient in exploration of the longitudinal motion in axial symmetric structures. For exploration of three-dimensional problems¡ however, by way of example we may take a uniformly charged sphere of radius a as a large-size particle. The formula for force of interaction of two such balls is known [8] and based on this formula, we can write the system of integro-differential equations [10] for modeling and optimization of beam dynamics of charged particles in three-dimensional case.

CONCLUSION

Considered integro-differential model for the dynamics of intense charged particle beams can be used effectively and is used in the solution of simulation and optimization of beam dynamics of charged particles in the accelerating and focusing structures.

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