THE STOCHASTIC CHARACTERISTICS STABILITY IN THE PROBLEM OF OBSERVATION AND ESTIMATE OF THE CHARGED PARTICLES MOVEMENT

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Abstract

The charged beam moving in the accelerator is modelled by particle-in-cell method. The control with delay in the connection canal is simulated by the after-effect. Thus, the model is described by system of the differential equations with delay. We observe and estimate changes of particles coordinates in the cross-section of the accelerator. It is supposed that initial conditions are the set with random error and we have chance of updating the solution in periodic timepoints with the same error. The dependence between the estimate dispersion and the measuring error dispersion is recieved.

INTRODUCTION

Consider a beam of charged particles, moving in the accelerator. We research them by particle-in-sell method. Suppose that the problem is to calculate the coordinates in the transverse section of the accelerator.

We'll linearize the equations, describing one of the large particles movement. Herewith we add a delay in the system. In such way we'll simulate the control with delay in the connection canal [1].

Thus the dynamic of the beam coordinates will be described by the linear equations system with delay:

$$\dot{z}(t) = Az(t) + Bz(t-h), \tag{1}$$

where A, B nonsingular 2x2 constant matrices,

$$z(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

is the vector of the beam coordinates in the cross-section of the accelerator. The solution of the equation is uniquely determined by the initial function $\varphi(t)$, defined on the interval $[-h; 0]$.

Suppose we can measure the initial position of the beam with some error. We'll designate by $\hat{z}(t)$ the solution with measured initial values $\hat{\varphi}(t)$, $t \in [-h; 0]$, and consider its difference with the exact solution.

$$\bar{z}(t) = z(t) - \hat{z}(t)$$

The value of the error is obviously depending on the measuring error of the initial value, which is equal

$$\bar{\varphi}(t) = \varphi(t) - \hat{\varphi}(t)$$

We suppose that the measuring error is a centered random variable and that the errors of the components measuring aren't depending on each other. Thus, $\bar{\varphi}(t)$ on $[-h; 0]$ is a centered random vector with the covariation matrix $\sigma^2 I$.

Now we'll build the upper limit of the covariation matrix $D(\bar{z}(t))$ as a function of t for $t > 0$. We note, that for each A, B, z there is

$$\left\|\sqrt{AA^{\mathrm{T}}}z\right\| \geq \|Az\|, \left\|\sqrt{BB^{\mathrm{T}}}z\right\| \geq \|Bz\|.$$

Because of the positive definiteness of the matrices $\sqrt{AA^{\mathrm{T}}}$ and $\sqrt{BB^{\mathrm{T}}}$, if we put them into (1) instead of A, B, then each component $\dot{z}(t)$ will have the same sign as the corresponding component $z(t)$, and it means that the module of each component is monotonicaly increasing on $t > 0$, and $\|z(t)\|$ is increasing too. Thus, if we change the equation (1) with

$$\dot{w}(t) = \left(\sqrt{AA^{\mathrm{T}}} + \sqrt{BB^{\mathrm{T}}}\right)w(t), \tag{2}$$

we get the function, which is the upper limit of the solution of the system (1). At each moment it will be a random vector, and its covariation matrix norm will be larger than the covariation matrix norm of the system (1) solution.

OBSERVATION WITH CORRECTION

Consider the set of the moments $n\tau$, then the solution of (2) at these moments is given by the formula $w(n\tau) = \Phi_n w(0)$, where $\Phi_n = e^{\left(\sqrt{AA^{\mathrm{T}}}+\sqrt{BB^{\mathrm{T}}}\right)n\tau}$.

The covariation matrix of $w(n\tau)$ is:

$$D\big(w(n\tau)\big) = \Phi_n \sigma^2 I \Phi_n^{\mathrm{T}} = \sigma^2 \Phi_n \Phi_n^{\mathrm{T}} = \sigma^2 \Phi_n^2$$

because of the symmetry of the matrix Φ_n. Because of the positive definiteness of the matrix Φ_n^2 matrix its spectral norm is equal its largest eigenvalue $e^{2\lambda n\tau}$, where λ is the largest eigenvalue of the matrix $\sqrt{AA^{\mathrm{T}}} + \sqrt{BB^{\mathrm{T}}}$. If we want to bound the error dispersion of $\bar{z}(t)$, we can bound the function $M(t) = \big\|D\big(w(t)\big)\big\|$.

Suppose on each segment $[n\tau - h, n\tau]$, $\tau = const$, $\tau > h$, $n = 1,2,\ldots$, we can measure the values of $z(t)$ with the error dispersion σ^2. Then we have 2-dimensional case of the dynamic filtration, described in [2].

Suppose this correction will be made not surely, but with the probability, which means that $M(t)$ will be a stochastic function. At each moment $n\tau$ we can calculate its expectation value $E\big(M(n\tau)\big)$ and the dispersion $D\big(M(n\tau)\big)$. Thus, the function $M(t)$ is left-continuous function and it increases on each segment $(n\tau, (n+1)\tau]$, and then it begin to increase from the value σ^2.

Let p is chance of correction response at any moment $n\tau$, $n = 1,2,\ldots$ Denote $q = 1 - p$, $E_n = E\big(M(n\tau)\big)$, $D_n = D\big(M(n\tau)\big)$, $Y_n = \sigma^2 e^{2\lambda n\tau}$. It is obvious that the set of possible values E_n is $\{Y_i\}_{i=1}^{n}$.

STOCHASTIC CHARACTERISTECS STABILITY

Lemma 1

E_n is monotonically increasing by n.

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Proof

Notice that expectancy $P\{M(n\tau - 0) = Y_i\}$ is $q^{i-1}p$ for $i = \overline{1, n-1}$ and it equals q^{i-1} for $i = n$.

Therefore,

$$E_n = q^{n-1}Y_n + \sum_{i=1}^{n-1} q^{i-1}p\, Y_i$$

$$= q^{n-1}\sigma^2 e^{2\lambda n\tau} + p\sigma^2 e^{2\lambda\tau} \sum_{i=0}^{n-2} \left(qe^{2\lambda\tau}\right)^i$$

Hence,

$$E_{n+1} - E_n = \sigma^2 \Big(q^n e^{2\lambda(n+1)\tau} - q^{n-1}e^{n\alpha\tau} + pe^{2\lambda\tau}\left(qe^{2\lambda\tau}\right)^{n-1}\Big)$$

$$= \sigma^2 \left(q^n e^{2\lambda(n+1)\tau} - e^{2\lambda n\tau}(-q^{n-1} + pq^{n-1})\right).$$

Having regard to $q^{n-1} + pq^{n-1} = q^n$ and $\alpha\tau > 0$, we arrive to

$$E_{n+1} - E_n = \sigma^2 q^n e^{2\lambda n\tau}\left(e^{2\lambda\tau} - 1\right) > 0. \blacksquare$$

Lemma 2

D_n is monotonically increasing by n.

Proof

We have

$D_n = \sum_{i=1}^{n-1} q^{i-1}p\,(Y_i - E_n)^2 + q^{n-1}(Y_n - E_n)^2$,

hence

$$D_{n+1} - D_n = \sum_{i=1}^{n} q^{i-1}p\,(Y_i - E_{n+1})^2 + q^n(Y_{n+1} - E_{n+1})^2 - \sum_{i=1}^{n-1} q^{i-1}p\,(Y_i - E_n)^2 - q^{n-1}(Y_n - E_n)^2$$

$$= \sum_{i=1}^{n-1} q^{i-1}p\,(Y_i - E_n)^2 + q^{n-1}(Y_n - E_n)^2$$

$$+ \sum_{i=1}^{n} q^{i-1}p\ \Delta E^2 - 2\sum_{i=1}^{n} q^{i-1}p(Y_i - E_n)\Delta E$$

$$+ q^n(Y_{n+1} - E_{n+1})^2 - \sum_{i=1}^{n-1} q^{i-1}p\,(Y_i - E_n)^2$$

$$- q^{n-1}(Y_n - E_n)^2$$

Since

$$2\Delta E\left[\sum_{i=1}^{n} q^{i-1}p(Y_i - E_n) + q^{n-1}(Y_n - E_n)\right] = 0,$$

and

$$\sum_{i=1}^{n} q^{i-1}p\ \Delta E^2 = \Delta E^2(1 - q^n),$$

where $\Delta E = E_{n+1} - E_n$, we have

$$D_{n+1} - D_n = \Delta E^2(1 - q^n) + q^n\Delta E^2 - 2q^n(Y_{n+1} - E_n)\Delta E + q^n(Y_{n+1} - E_n)^2 + q^{n-1}p(Y_n - E_n)^2$$

$$+ 2\Delta E q^n(Y_n - E_n) - q^{n-1}(Y_n - E_n)^2$$

$$= q^n[(Y_{n+1} - E_n)^2 - (Y_n - E_n)^2]$$

$$- 2q^n\Delta E[(Y_{n+1} - E_n) - (Y_n - E_n)] + \Delta E^2$$

$$= q^n(Y_{n+1} - Y_n)(Y_{n+1} + Y_n - 2E_n) - \Delta E^2$$

$$= \Delta E(Y_{n+1} + Y_n - 2E_n - \Delta E)$$

$$> \Delta E(Y_{n+1} - E_n - \Delta E) = \Delta E(Y_{n+1} - E_{n+1}) > 0. \blacksquare$$

Theorem

The sufficient condition for the stability of the solutions $E\big(M(t)\big) = 0$ and $D\big(M(t)\big) = 0$ for $\sigma^2 = 0$ on $[0, \infty)$ is the inequality $4\lambda\tau < \ln\frac{1}{q}$.

Proof

For any t exists such n that $E\big(M(n\tau)\big) > E\big(M(t)\big)$ and $D\big(M(n\tau)\big) > D\big(M(t)\big)$. That's why we consider only the moments $n\tau$.

If $2\lambda\tau < \ln\frac{1}{q}$, then $\lim_{n\to\infty} E\big(M(n\tau)\big) = \frac{\sigma^2 pe^{2\lambda\tau}}{1-qe^{2\lambda\tau}} = \bar{E}$.

If $4\lambda\tau < \ln\frac{1}{q}$, then

$\lim_{n\to\infty} D\big(M(n\tau)\big) = \sigma^4\left(\frac{pe^{4\lambda\tau}}{1-qe^{4\lambda\tau}} - \frac{p^2e^{4\lambda\tau}}{\left(1-qe^{2\lambda\tau}\right)^2}\right) = \bar{D}$, as it was shown in [3].

According to Lemma 1 and Lemma 2, $E\big(M(t)\big) < \bar{E}$ and $D\big(M(t)\big) < \bar{D}$. That's why for any ε if $\sigma^2 < \min\left\{\frac{\varepsilon}{\bar{E}}, \sqrt{\frac{\varepsilon}{\bar{D}}}\right\}$, then $E\big(M(t)\big) < \varepsilon$ and $D\big(M(t)\big) < \varepsilon$. $\blacksquare$

Example

In Fig. 1 we see the example of concrete realization of stochastic process for values $\sigma^2 = 1$, $\tau = 1$, $\lambda = 1/2$. Correction works at points $t = 1$, $t = 3$.

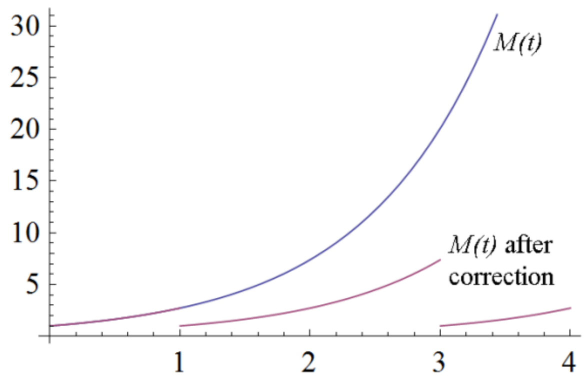

Figure 1: Plot of $M(t)$ before and after correction.

CONCLUSION

We obtain the sufficient conditions of the stochastic characteristics stability for system (1). If this inequality is satisfied, then our estimate is not far from the exact solution.

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