COMPLEX SHUNT IMPEDANCE AND BEAM-RF CAVITY INTERACTION

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Abstract

Two approaches usually are used to describe beam-cavity interaction in accelerator based applications. The first one is electro dynamical and uses Maxwell equations to derive appropriate equations, field modes expressions being necessary to calculate field amplitudes excited by moving charges in the cavity. The other one uses LC circuit to derive appropriate equations for voltage amplitude induced in cavity by accelerated bunches, thin accelerating gap to some extent being not fully correctly defined representation in such approach. In this paper, the expressions are derived that describe beam-RF cavity interactions in terms of so called complex shunt impedance, strict electro dynamical approach being used in calculations. It is shown that complex shunt impedance module coincides completely with usual shunt impedance definition that up to now is used widely to describe rf cavity efficiency. The physical sense of its phase is given in the paper as well. Both complex shunt impedance module and its phase can be calculated or measured experimentally.

INTRODUCTION

To analyze the processes resulting from beam-cavity interaction two approaches are used mainly. The first one is based on Maxwell equations solving. Cavity eigen functions for vector potential are found that together with differential equations for fields amplitudes form the basis for following analysis. In other approach mentioned the RF cavity is replaced with the electrical circuit containing active resistance, capacitance and inductance, their values are chosen in such a way to have the resonance frequency, quality factor and shunt impedance the same for the RF cavity and for the circuit. In this approach one has an analytical representation so necessary for analysis but the questions concerning approach justification and some uncertainness arise.

In this paper we use strict field approach based on Maxwell equation to derive the equation for field amplitude that might be suitable for processes analysis in accelerator containing RF cavity. Complex shunt impedance concept have been introduced and this appeared be fruitful for beam-cavity interaction processes description in RF accelerator based applications problems.

ELECTRODYNAMICS OF RF CAVITY-BEAM INTERACTION

To find out the fields that induces moving charge in a RF cavity, we will use the method that had been developed in [1]. Vortex electrical $\vec{E}(\vec{r},t)$ and magnetic $\vec{H}(\vec{r},t)$ fields are represented as derivatives of vector potential $\vec{A}(\vec{r},t)$ on time t and space $\vec{r}$ coordinates:

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t}, \quad \vec{H}(\vec{r},t) = \frac{1}{\mu_0} rot\vec{A}(\vec{r},t) \quad (1)$$

where μ_0 is magnetic permeability of free space. Here and later SI units are used. Vector potential satisfies the wave equation

$$\Delta\vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0 \vec{j}(\vec{r},t) \quad (2)$$

$\vec{j}(\vec{r}.t)$, c being current density and the light velocity.

To find out the expressions for vector potential we will use the most direct way. Namely, we represent vector potential as an expansion on the infinite sum of RF cavity eigen functions $\vec{A}_\lambda(\vec{r})$ with time dependent coefficients $g_\lambda(t)$:

$$\vec{A}(\vec{r},t) = \sum_{\lambda=1}^{\infty} g_\lambda(t)\vec{A}_\lambda(\vec{r}) \quad (3)$$

with the boundary conditions $\vec{A}_\lambda |_\Sigma = 0$ on cavity surface.

Starting from the equation (2) and taking into account (3) one can easily obtain the equations for cavity vector eigen functions and appropriate time dependent coefficients (fields amplitudes):

$$\Delta\vec{A}_\lambda(\vec{r}) + k_\lambda{}^2 \vec{A}_\lambda(\vec{r}) = 0 \quad (4)$$

$$\frac{d^2 g_\lambda(t)}{dt^2} + \omega_\lambda{}^2 g_\lambda(t) = \int_V \vec{j}(\vec{r},t)\vec{A}_\lambda(\vec{r})dV \quad (5)$$

Here $k_\lambda = \omega_\lambda / c$ are eigen values of boundary values problems (4), the specific solutions for RF cavities are called cavity modes, ω_λ being the eigen angular frequencies of appropriate modes, c is light velocity. Integration in formula is assumed to be performed over cavity volume V . Last equation can be generalized up to the next one

$$\frac{d^2 g_\lambda(t)}{dt^2} + \frac{\omega_\lambda}{Q_\lambda}\frac{dg_\lambda}{dt} + \omega_\lambda{}^2 g_\lambda(t) = \int_V \vec{j}(\vec{r}.t)\vec{A}_\lambda(\vec{r})dV \quad (6)$$

if losses in cavity and outside are taking into account. Here Q_λ stands for cavity quality factor:

$$Q_\lambda = \frac{\omega_\lambda W_\lambda}{P_\lambda} \quad (7)$$

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where W_λ is the electromagnetic energy in the mode λ, stored in cavity volume and P_λ represents the total RF power losses that besides ohm losses in cavity walls includes the external losses due to cavity coupling with external circuits. It is supposed that eigen functions are normalized by the condition

$$\int_V A_\lambda^2 = \mu_0 c^2 = 1/\varepsilon_0 \quad (8)$$

Here μ_0 and ε_0 are magnetic and electric permeability respectively.

For the analysis followed we will use the cavity excitation equation in the form with small RF losses, and this has no any influence on generality of results to be obtained. Then, all calculations will be made for a single charged particle with charge value q of zero dimensions in all directions entering cavity at moment $t = 0$. In such a case the total current density

$$\vec{j}(\vec{r},t) = q\vec{v}(\vec{r},t)\delta(x, y, vt)), \quad (9)$$

where $\vec{v}(\vec{r},t)$ stands for particle velocity being assumed constant within the cavity, and $\delta()$ is Dirac delta function: $\delta(x) = \infty$ for $x = 0$, $\delta(x) = 0$ for $x \neq 0$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \quad (10)$$

We suppose also the case that is the most interesting for accelerator based applications – the particle moves along cavity axis where

$$x = 0, y = 0, z = vt \quad (11)$$

In such assumptions:

$$\frac{d^2 g(t)}{dt^2} + \omega^2 g(t) = \int_0^L \delta(z - vt) qvA(z)dz \quad (12)$$

From here and to the paper end we omit mode indexes that does not lead to ambiguity. It follows from last relation that

$$\frac{d^2 g(t)}{dt^2} + \omega^2 g(t) = J(t), \; J(t) = qvA(vt)\eta(t)\eta(L - vt) \quad (13)$$

were $A(z) = A_z(0,0,z)$ and $\eta()$ is Heaviside step function

$\eta(x) = 1$ for all $x \geq 0$, $\eta(x) = 0$ for all $x < 0$ (14)

The solution of the equation (13) that satisfies initial conditions $g(0) = \dot{g}(0) = 0$ (corresponding equal to zero electric and magnetic components of induced field) can be represented in the form [2]:

$$g(t) = \frac{1}{\omega}\int_0^{L/v} J(\tau)\sin\omega(t-\tau)d\tau = \frac{\sin\omega t}{\omega}J_1 - \frac{\cos\omega t}{\omega}J_2 \quad (14)$$

Here

$$J_1 = \int_0^{L/v} J(\tau)\cos\omega\tau\, d\tau, \quad J_2 = \int_0^{L/v} J(\tau)\sin\omega\tau\, d\tau \quad (15)$$

Note that solution for field amplitude in the form (14) is valid for time interval $t > L/v$.

INDUCED VOLTAGE OVER CAVITY EXTERNAL PARAMETERS

Let us find out probe particle with charge e energy gain after passage of the cavity assuming field amplitude being $g(t) = a\sin(\omega t + \varphi)$, where a is constant. One can derive easily:

$$\mathrm{E}(\varphi) = -e\int \dot{g}(t)A(z)dz = -\frac{ea\omega}{q}\cos\varphi J_1 + \frac{ea\omega}{q}\sin\varphi J_2 \quad (16)$$

Representing rf cavity in the form of equivalent thin gap of zero length (accelerating gap) with applied rf voltage one can conclude that appropriate voltage amplitude U_m is equal to

$$U_m = \frac{\mathrm{E}}{e} = \frac{a\omega}{q}\left(J_1^2 + J_2^2\right)^{1/2} \quad (17)$$

This can be expressed in terms of cavity shunt impedance R and cavity quality factor Q:

$$R = \frac{U_m^2}{P_0}, \; Q_0 = \frac{\omega W}{P_0} \quad (18)$$

where P_0 stands for cavity walls power losses and W is electromagnetic energy stored in the cavity volume.

$$W = \frac{\varepsilon_0}{2}\int_V E_m^2 dV = \frac{a^2\omega^2\varepsilon_0}{2}\int_V A^2(\vec{r})dV \quad (19)$$

Taking into account normalization condition one arrives finally at relations

$$W = \frac{a^2\omega^2}{2}, \; J_1^2 + J_2^2 = \frac{R}{Q_0}\frac{\omega q^2}{2} \quad (20)$$

Let us calculate energy loss for the particle traversing cavity filled with the field induced by previous charge, both radiating charge and probe particle being spaced by time interval equal to period of rf oscillations.

$$\mathrm{E}_{lost} = -ev\int_0^{L/v} \dot{g}(t)A(vt)dt = -ev\int_0^{L/v}\left[\int_0^{L/v} J(\tau)\cos(\omega t - \omega\tau)d\tau\right]A(vt)dt = -\frac{e}{q}\left(J_1^2 + J_2^2\right) \quad (21)$$

Together with last expression this gives

$$\mathrm{E}_{lost} = -\frac{eq\omega}{2}\frac{R}{Q_0} \quad (22)$$

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In terms of thin gap this means that bunch with charge q induces rf voltage of amplitude

$$U = \frac{q\omega}{2}\frac{R}{Q_0} \quad (23)$$

and rf phase π. Furthermore, taking into account field damping we arrive finally to the next expression for rf field, induced by charged bunch on equivalent thin gap

$$U = -\frac{q\omega}{2}\frac{R}{Q_0}\exp(-\omega t/2Q)\cos\omega t \quad (24)$$

Very often, current value I averaged over RF period is used instead of charge value

$$U = -\pi I\frac{R}{Q_0}\exp(-\omega t/2Q)\cos\omega t \quad (25)$$

RF CAVITY COMPLEX SHUNT IMPEDANCE CONCEPT

It follows from (14) that the phase of oscillating depends on two quantities J_1 and J_2, and these two quite different functionals can not be expressed over one quantity. These parameters might be used for detailed description of beam-cavity interaction and the outlook on relation (14) prompts to represent it in the form

$$g(t) = \frac{\sin\omega t}{\omega}J_1 - \frac{\cos\omega t}{\omega}J_2 = \frac{D}{\omega}\left(\sin\omega t\sin\psi - \cos\omega t\cos\psi\right) \quad (26)$$

where

$$D = \sqrt{J_1^2 + J_2^2}\ ,\ \sin\psi = \frac{J_1}{D}, \cos\psi = \frac{J_2}{D} \quad (27)$$

and formula takes the form:

$$g(t) = -\frac{D}{\omega}\cos(\omega t + \psi) \quad (28)$$

Thus, the pare of quantities J_1 and J_2 or D and ψ is needed for detailed description of beam-cavity interaction, and this pare, as it followes from formulae written above, might be considered as the real and imaginary parts or the module and the phase of complex quantity:

$$\hat{D} = \operatorname{Re} D + i\operatorname{Im} D = D\exp i\psi \quad (29),$$

where i is imaginary unit. D is expressed over cavity shunt impedance, and finally expression for field amplitude looks like

$$g(t) = -q\sqrt{\frac{R}{2\omega Q_0}}\cos(\omega t + \psi) \quad (30)$$

It is often much more convenient to deal with complex quantities remembering that physical sense has its real part. Then, denoting

$$Z = R\exp(i2\psi)\ , \quad (31)$$

we arrive at relation

$$g(t) = -\operatorname{Re} q\sqrt{\frac{Z}{2\omega Q_0}}\exp i\omega t \quad (32)$$

In these notations, it is quite natural to refer to Z as complex shunt impedance. Its module coincides with usual cavity shunt impedance. To establish physical sense its phase let us rewrite expression (16) for energy gain for the probe particle entering a cavity at $t = 0$ using ψ definition:

$$U(\varphi) = \frac{a\omega}{qv}D(-\cos\varphi\sin\psi + \sin\varphi\cos\psi) = \frac{a\omega}{qv}D\sin(\varphi - \psi) \quad (33)$$

Complex cavity shunt impedance can be calculated for any particular mode according formulae above or established experimentally. To measure R and ψ the following experiment has to be done. RF cavity installed on probe beam path is fed with power P. Cavity RF phase is adjusted to have the maximum energy gain U_m at its exit. Appropriate combination (18) of values obtained gives cavity shunt impedance module. Adjusting phase shifter to position corresponding zero energy gain at cavity exit one gets information concerning phase ψ .

CONCLUSION

The concept of complex shunt impedance has been introduced to the problem under attention, and solution for field amplitude had been expressed in terms of this cavity parameter. The physical sense both for the module and the phase as well of complex shunt impedance has been clarified. The first one is simply cavity shunt impedance in widely used sense, while the other fixes the phase at which the probe particle, entering cavity, traverses it without additional energy gain. It had been shown, that complex shunt impedance components can be calculated or measured experimentally.

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