SECOND ORDER METHOD FOR BEAM DYNAMICS OPTIMIZATION *

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Mathematical methods of beam dynamics optimization was developed in the works of D.A. Ovsyannikov (see [1]). These methods are based on numerical calculation of the first derivatives on accelerator structure parameters of functional estimating quality of a beam. They allow to find accelerator structures with satisfactory parameters and also to improve existing structures. The present paper is devoted to new method based on numerical calculation of the second derivatives of the functional. This method can be considered as an extension of the methods of first order.

BEAM DYNAMICS CONTROL PROBLEM

Consider a beam describing by the particle distribution density $\varrho(x)$ in the phase space Ω, $x \in \Omega$. Let at the initial moment t_0 the particle distribution density [2] is given on some p-dimensional surface $S : \varrho(t_0, x) = \varrho_{(0)}(x) = \varrho_{(0)\,1\ldots p}(x)\, dx^1 \wedge \ldots \wedge dx^p$, $p \le \dim \Omega$, where x^i, $i = \overline{1,p}$, are coordinates on S_0 which can be taken also as some of coordinates in the phase space.

Assume that the particle trajectories are described by the differential equation

$$\frac{dx}{dt} = f(t, x, u),$$

where t is trajectory parameter, $t \in [t_0, T]$, u is control function, $u(t) \in U \subset R^r$. Assume that vector f is defined in a domain $[t_0, T] \times \Omega \times U$, and that the solution of the Cauchy problem for this equation with initial condition $x(t_0) = x_0$ uniquely exists for any x_0 under consideration.

Let introduce functional characterizing quality of the controlled process

$$\Phi(u) = \int\limits_{\Omega} g(x_T)\varrho(T, x_T), \qquad (1)$$

where $g(x)$ is a piecewice continuous function, and integral on Ω means in fact integration over image of initial surface S_0 of corresponding differential form satisfying to the Vlasov equation [2]. The problem of minimizing of functional (1) on control function u from U is called the terminal problem of beam control with account of particle distribution density.

METHOD FORMULATION

Equation for the first variation of x has the form

$$\frac{d\delta x^i}{dt} = \frac{\partial f^i}{dx^j}\delta x^j + \delta_u f^i, \qquad \delta x^i(t_0) = 0, \qquad (1)$$

where

$$\delta_u f^j = \frac{\partial f^j}{\partial u^k}\delta u^k$$

(summation is meant on coincident indices). The solution of the problem (1) can be written as

$$\delta x^i(t) = \int\limits_{t_0}^{t} G^i_j(t, t')\delta_u f^j(t')dt',$$

where $G(t, t')$ is the Green matrix of the system (1), satisfying to the equation

$$\frac{dG^i_j(t, t')}{dt} = \frac{\partial f^i}{\partial x^k} G^k_j(t, t'),$$

and to the condition $G(t, t) = E$, where E is identity matrix.

Then variation of the functional (1) can be written in the form

$$\delta_u \Phi = \int\limits_{t_0}^{T} \int\limits_{\Omega} \frac{\partial g}{\partial x} G(T, t')\delta_u f(t, x)\varrho(t, x)\, dt. \qquad (1)$$

Let introduce the differential form

$$\psi(t, x) = -\frac{\partial g}{\partial x}|_{x=x_T} G(T, t),$$

satisfying to equation and condition

$$\frac{d\psi}{dt} = -\psi\frac{\partial f}{\partial x}, \qquad \psi(T) = -\frac{\partial g}{\partial x}|_{x=x_T}.$$

Then the functional variation (1) takes the form

$$\delta_u \Phi = -\int\limits_{t_0}^{T} \int\limits_{\Omega} \psi(t, x)\delta_u f(t, x)\varrho(t, x)\, dt.$$

Assume that u is a piecewise constant vector function

$$u = u_i, \quad t \in [t_{i-1}, t_i), \quad i = \overline{1, M}, \quad t_M = T.$$

Then fuctional (1) can be considered as function of rM control parameters. The derivatives on these parameters are

$$\frac{\partial \Phi}{\partial u^k_i} = -\int\limits_{t_0}^{T} \int\limits_{\Omega} \psi(t, x)\frac{\partial \delta_u f(t, x)}{\partial u^k_i}\varrho(t, x)\, dt. \qquad (1)$$

Passing to the summation on macroparticles within the framework of the method of macroparticles, write the functional derivatives in the form

$$\frac{\partial \Phi}{\partial u^k_i} = -\int\limits_{t_0}^{T} \sum_{j=1}^{N} \psi(t, x_{(j)}))\frac{\partial \delta_u f(t, x_{(j)})}{\partial u^k_i}\, dt,$$

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where $x_{(j)}$ denotes position in the phase space of the j-th particle.

Consider second derivatives of the functional on the control parameters. For simplicity assume that $r = 1$ (one scalar control function). Let us consider second derivatives only on the same parameters $\partial^2\Phi/\partial u_i^2$. As

$$\frac{\partial x^j}{\partial u_i}(t) = \int\limits_{t'}^{t} G_k^j(t,t')\frac{\partial \delta_u f^k}{\partial u_i}(t')\,dt', \qquad (1)$$

the expression (1) can be rewriteen in the form

$$\frac{\partial \Phi}{\partial u_i} = \int\limits_{\Omega} \frac{\partial \Phi}{\partial x^j}\frac{\partial x^j}{\partial u_i}(T)\varrho(T). \qquad (1)$$

Assume also that $\partial^2\Phi/(\partial x^i \partial x^j) = 0$ if $i \neq j$. Then

$$\frac{\partial^2 \Phi}{\partial u_i^2} = \int\limits_{\Omega} \varrho(T)\left[\frac{\partial \Phi}{\partial x^j}\frac{\partial^2 x^j}{\partial u_i^2}(T) + \frac{\partial^2 \Phi}{\partial (x^j)^2}[\frac{\partial x^j}{\partial u_i}(T)]^2\right].$$

Passing to the summation on macroparticles we get

$$\frac{\partial^2 \Phi}{\partial u_i^2} = \sum_{k=1}^{N}\left[\frac{\partial \Phi}{\partial x^j}\frac{\partial^2 x^j_{(k)}}{\partial u_i^2}(T) + \frac{\partial^2 \Phi}{\partial (x^j)^2}[\frac{\partial x^j_{(k)}}{\partial u_i}(T)]^2\right],$$

where the first derivatives are expressed by (1).

It can be shown that when f^j are linear on control parameters u_i, second variation of x has the form

$$\delta^2 x^j(t) = \int\limits_{t_0}^{t} (D^j_{kl}(t,t')\delta_u f^k(t') + G_k^j(t,t')\delta_u(\frac{\partial f^k}{\partial x^l})|_{t'})\times$$

$$\times(\int_{t_0}^{t'} G^l_m \delta_u f^m(t'')\,dt'')\,dt',$$

where components of the tensor D satisfy to the system of differential equations

$$\frac{\partial D^i_{lk}(t,t')}{\partial t'} = -2D^i_{lm}(t,t')\frac{\partial f^m}{\partial x^k}(t') + G^i_m(t,t')\frac{\partial^2 f^m}{\partial x^l \partial x^k}(t')$$

and the condition

$$D^i_{lk}(t,t) = 0, \quad i,j,k = \overline{1,m}.$$

Then

$$\frac{\partial^2 x^j}{\partial u_i^2}(t) = \int\limits_{t_{i-1}}^{t_i}\left[D^j_{kl}(t,t')\frac{\partial \delta_u f^k}{\partial u_i}(t') + \right.$$

$$\left. + G^j_k(t,t')\frac{\partial}{\partial u_i}\big(\delta_u(\frac{\partial f^k}{\partial x^l})\big)|_{t'}\right]\times$$

$$\times\int\limits_{t_{i-1}}^{t'} G^l_m(t',t'')\frac{\partial \delta_u f^m}{\partial u_i}(t'')\,dt''\,dt'.$$

Numerical optimization process can be implemented as a sequence of steps of numerical calculation of first and second derivatives of the functional, and changing of control parameters according to the expression

$$\delta u_i = -\frac{\partial \Phi/\partial u_i}{\partial^2 \Phi/\partial u_i^2.} \quad i = \overline{1,M}$$

while the functional is decreasing. If at some step it will be turned out that $\partial^2\Phi/\partial u_i^2 = 0$ for some i, one should combine this method with method of gradient descent or another first order method.

OPTIMIZATION OF RFQ CANNEL

Assume that longitudinal component of electric field in the RFQ channel is

$$E_z = U_0\frac{4kT}{\pi}\cos\eta\cos\omega t, \quad \eta(z) = \int\limits_{z_0}^{z} k(z')\,dz', \quad (1)$$

Here $2U_0$ is intervane voltage, ω is frequency of the field oscillations, a is aperture of the cell, $k = \pi/L$, L is the cell length, which varies along the channel, $\eta(z)$ is the phase of electrode modulation, T is acceleration efficiency.

Within the framework of this model, the longitudinal motion does not depend of the transverse motion. It allows us to consider logitudinal motion separately. For simplicity, consider optimization problem accounting only longitudinal motion.

Take reduced energy γ and phase of the particle $\varphi = \omega t$ as the phase coordinates. Initial distribution in the phase space of longitudinal motion can be set in various manner. For example it can be taken in the form $\varrho_{(0)\,\varphi} = (2\pi)^{-1}$, $\varphi_0 \in [-2\pi, 0]$, $\gamma = \gamma_0$. Here $\varrho_{(0)\,\varphi}$ is $\varphi-$component of the initial distribution density, φ_0 and γ_0 are initial phase and energy of a particle.

Consider the difference between phase of the synchronous particle φ_s and the phase of space modulation η

$$\Phi_s = \varphi_s - \int \overline{k}\,d\zeta. \qquad (1)$$

Here $\zeta = z/\lambda$, $\overline{k} = \lambda k$, $\lambda = 2\pi c/\omega$. Take function $u_1(\zeta) = d\Phi_c/d\zeta$ as the first control function. Let T be the second control function: $u_2(\zeta) = T(\lambda\zeta)$.

The equation of longitudinal dynamics for low intensity beam can be written in the form

$$\frac{d\varphi}{d\zeta} = 2\pi\gamma(\gamma^2 - 1)^{-1/2}, \qquad (1)$$

$$\frac{d\gamma}{d\zeta} = C_L(2\pi\gamma_s(\gamma_s^2 - 1)^{-1/2} - u_1)u_2\cos\eta\cos\varphi, \quad (1)$$

where $C_L = 2eU_0/(\pi mc^2)$. Equation for η has form [3]

$$\frac{d\eta}{d\zeta} = 2\pi\gamma_s(\gamma_s^2 - 1)^{-1/2} - u_1.$$

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Then equations for ψ can be written in the form

$$\frac{d\psi_\eta}{d\zeta} = \sum_{i=1}^{N} \psi_{(i)\gamma} C_L \overline{k} u_2 \sin\eta \cos\varphi_i,$$

$$\frac{d\psi_{(s)\gamma}}{d\zeta} = \psi_\eta \frac{2\pi}{(\gamma_s^2 - 1)^{3/2}} +$$

$$+\psi_{(s)\varphi} \frac{2\pi}{(\gamma_s^2 - 1)^{3/2}} \sum_{i=1}^{N} \psi_{(i)\gamma} \frac{2\pi}{(\gamma_s^2 - 1)^{3/2}} C_L u_2 \cos\eta \cos\varphi_i,$$

$$\frac{d\psi_{(i)\varphi}}{d\zeta} = C_L\, \psi_{(i)\gamma} \overline{k} T \cos\eta \sin\varphi_i, \qquad i = \overline{1, N},$$

$$\frac{d\psi_{(i)\gamma}}{d\zeta} = \psi_{(i)\varphi} \frac{2\pi}{(\gamma_i^2 - 1)^{3/2}}, \qquad i = \overline{1, N}.$$

Here i is number of a macroparticle. It is written in parenthesis at ψ to avoid confuse with indices. Let control functions are constant inside cells: $u_i(\zeta) = u_{ij}$, $\zeta \in [\zeta_{j-1}, \zeta_j)$, $j = \overline{1, M}$. Then

$$\frac{\partial \Phi}{\partial u_{1j}} = \int_{\zeta_{j-1}}^{\zeta_j} (\psi_\eta + \sum_{i=1}^{N} \psi_{(i)\gamma} C_L u_2 \cos\eta \cos\varphi_i)\, d\zeta,$$

$$\frac{\partial \Phi}{\partial u_{2j}} = \int_{\zeta_{j-1}}^{\zeta_j} \sum_{i=1}^{N} \psi_{(i)\gamma} C_L \overline{k} \cos\eta \cos\varphi_i \, d\zeta.$$

Restrict ourselves to the case of one scalar control function $u = T$. Then equation for Green functions and for components of tensor D are

$$\frac{dG_\varphi^\varphi}{d\zeta} = -G_\gamma^\varphi C_L \overline{k} T \cos\eta \sin\varphi, \ \frac{dG_\gamma^\varphi}{d\zeta} = G_\varphi^\varphi \frac{2\pi}{(\gamma^2 - 1)^{3/2}},$$

$$\frac{dG_\varphi^\gamma}{d\zeta} = -G_\gamma^\gamma C_L \overline{k} T \cos\eta \sin\varphi, \ \frac{dG_\gamma^\gamma}{d\zeta} = G_\varphi^\gamma \frac{2\pi}{(\gamma^2 - 1)^{3/2}},$$

$$\frac{\partial D_{\varphi\varphi}^\varphi(\zeta, \zeta')}{\partial \zeta'} = (2D_{\varphi\gamma}^\varphi \sin\varphi - G_\gamma^\varphi \cos\varphi) C_L \overline{k} T \cos\eta,$$

$$\frac{\partial D_{\varphi\varphi}^\gamma(\zeta, \zeta')}{\partial \zeta'} = (2D_{\varphi\gamma}^\gamma \sin\varphi - G_\gamma^\gamma \cos\varphi) C_L \overline{k} T \cos\eta,$$

$$\frac{\partial D_{\varphi\gamma}^\varphi(\zeta, \zeta')}{\partial \zeta'} = -\frac{4\pi D_{\varphi\varphi}^\varphi}{(\gamma^2 - 1)^{3/2}} + 2D_{\gamma\gamma}^\varphi C_L \overline{k} T \cos\eta \sin\varphi,$$

$$\frac{\partial D_{\varphi\gamma}^\gamma(\zeta, \zeta')}{\partial \zeta'} = -\frac{4\pi D_{\varphi\varphi}^\gamma}{(\gamma^2 - 1)^{3/2}} + 2D_{\gamma\gamma}^\gamma C_L \overline{k} T \cos\eta \sin\varphi,$$

$$\frac{\partial D_{\gamma\gamma}^\varphi(\zeta, \zeta')}{\partial \zeta'} = -\frac{4\pi D_{\varphi\gamma}^\varphi}{(\gamma^2 - 1)^{3/2}} - G_\varphi^\gamma C_L \overline{k} T \cos\eta \cos\varphi,$$

$$\frac{\partial D_{\gamma\gamma}^\gamma(\zeta, \zeta')}{\partial \zeta'} = -\frac{4\pi D_{\varphi\gamma}^\gamma}{(\gamma^2 - 1)^{3/2}} - G_\gamma^\gamma C_L \overline{k} T \cos\eta \cos\varphi.$$

Second derivatives of the functional are $\partial^2\Phi / \partial T_j^2 =$

$$= \sum_{i=1}^{N} \Big\{ \frac{\partial^2 \Phi}{\partial \varphi^2} \Big[\int_{\zeta_{j-1}}^{\zeta_j} G_\gamma^\varphi(\zeta_M, \zeta') \overline{k} \cos\eta(\zeta') \cos\varphi(\zeta')\, d\zeta' \Big]^2 +$$

$$+ \frac{\partial^2 \Phi}{\partial \gamma^2} \Big[\int_{\zeta_{j-1}}^{\zeta_j} G_\gamma^\gamma(\zeta_M, \zeta') \overline{k} \cos\eta(\zeta') \cos\varphi(\zeta')\, d\zeta' \Big]^2 +$$

$$+ \frac{\partial \Phi}{\partial \varphi} \int_{\zeta_{j-1}}^{\zeta_j} D_{\gamma\varphi}^\varphi(\zeta_M, \zeta') \overline{k} \cos\eta(\zeta') \cos\varphi(\zeta') \times$$

$$\times \Big[\int_{\zeta_{j-1}}^{\zeta'} G_\gamma^\varphi(\zeta', \zeta'') \overline{k} \cos\eta(\zeta'') \cos\varphi(\zeta'')\, d\zeta'' \Big]\, d\zeta'$$

$$+ \frac{\partial \Phi}{\partial \varphi} \int_{\zeta_{j-1}}^{\zeta_j} D_{\gamma\gamma}^\varphi(\zeta_M, \zeta') \overline{k} \cos\eta(\zeta') \cos\varphi(\zeta') \times$$

$$\times \Big[\int_{\zeta_{j-1}}^{\zeta'} G_\gamma^\gamma(\zeta', \zeta'') \overline{k} \cos\eta(\zeta'') \cos\varphi(\zeta'')\, d\zeta'' \Big]\, d\zeta'$$

$$+ \frac{\partial \Phi}{\partial \gamma} \int_{\zeta_{j-1}}^{\zeta_j} D_{\gamma\varphi}^\gamma(\zeta_M, \zeta') \overline{k} \cos\eta(\zeta') \cos\varphi(\zeta') \times$$

$$\times \Big[\int_{\zeta_{j-1}}^{\zeta'} G_\gamma^\varphi(\zeta', \zeta'') \overline{k} \cos\eta(\zeta'') \cos\varphi(\zeta'')\, d\zeta'' \Big]\, d\zeta'$$

$$+ \frac{\partial \Phi}{\partial \gamma} \int_{\zeta_{j-1}}^{\zeta_j} D_{\gamma\gamma}^\gamma(\zeta_M, \zeta') \overline{k} \cos\eta(\zeta') \cos\varphi(\zeta') \times$$

$$\times \Big[\int_{\zeta_{j-1}}^{\zeta'} G_\gamma^\gamma(\zeta', \zeta'') \overline{k} \cos\eta(\zeta'') \cos\varphi(\zeta'')\, d\zeta'' \Big]\, d\zeta' \Big\} C_L^2.$$

Particle number i at G, D, φ, γ is omitted for brevity. Analogous expressions can be obtained in 6-dimensional general case with 3 control functions [3].

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