



MATHEMATICAL MODELING AND OPTIMIZATION OF BEAM DYNAMICS IN ACCELERATORS

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Mathematical methods of modeling and optimization are extensively used in many fields of science and technology. Development of specialized software for various applications is of more and more importance. A special class of tasks attracting attention of numerous researches includes the problems associated with the beam dynamics optimization in accelerator. There are not the general methods of accelerating and focusing structures optimization. However as the demand to output beam parameters are progressively increasing it is needed to develop a new approaches and methods to solve these problems.

In the paper the different mathematical control models describing beam dynamics are presented. Especially we consider the problems related to charged particles interaction. In this case we investigate the controlled dynamic process described by a system of integro - differential equations. The optimization methods are developed for the different functionals concerned with the quality of beam. They are used for solution of various beam dynamics problems in linac . In particular we investigate the optimization problem of a radial matching section in RFQ channel. We consider the problem of construction self-consistent distribution for charged particle beam in magnetic field too.





Mathematical control models

Mathematical description (model) of dynamical process — equations of motion.

Choice of control functions — parameters of dynamical object, which should be defined on the design stage.

Functionals of quality that characterize dynamics and output parameters (characteristics) of the structure to be optimized.





Mathematical Optimization Models (MOM)

MOM 1.

$$\frac{dx}{dt} = f(t, x, u) \tag{1}$$

$$I(u) = \int_{0}^{T} \int_{M_{t,u}} \varphi(t, x_{t}, u(t, x_{t})) dx_{t} dt + \int_{M_{t,u}} g(x_{t}) dx_{t} \rightarrow \min_{u \in D}$$
$$M_{t,u} = \{ x_{t} = x(t, x_{0}, u), x_{0} \in M_{0}, x(0) = x_{0} \}$$

(3)



MOM 2.

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(4)

$$\frac{dx}{dt} = f(t, x, u), \quad x(0) \in M_0$$
(1)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} f(t, x, u) + \rho \operatorname{div}_{x} f(t, x, u) = 0$$
(2)

 $t \in [0,T], x \in \mathbb{R}^n, u \in U \subset \mathbb{R}^r$

$$\rho(0,x) = \rho_0(x), \ \rho = \rho(t,x)$$
(3)

$$I(u) = \int_{0}^{T} \int_{M_{t,u}} \varphi(t, x_t, \rho(t, x_t), u) dx_t dt +$$
$$+ \int_{M_{t,u}} g(x_T, \rho_T(T, x_T)) dx_T \rightarrow \min_{u=u(t) \in D}$$

$$div_{x}f = \sum_{i=1}^{n} \frac{\partial f_{i}(t, x, u)}{\partial x_{i}}$$



MOM 3.

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$$\frac{dx}{dt} = f_1(t, x, u) + \int_{M_{t,u}} f_2(t, x, y_t) \rho(t, y_t) dy_t =$$

$$= f(t, x, u), x(0) = x_0 \in M_0 \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} f(t, x, u) + \rho div_x f(t, x, u) = 0 \quad (2)$$

$$\rho(0, x) = \rho_0(x), \rho = \rho(t, x)$$

$$I(u) = \int_{0}^{T} \int_{M_{t,u}} \phi(t, x_t, \rho(t, x_t), u) dx_t dt +$$

$$+ \int_{M_{t,u}} g(x_T, \rho_T(T, x_T)) dx_T \to \min_{u=u(t) \in D} \quad (3)$$

 f_l is vector function defined by external electromagnetic fields,

 f_2 is vector function determining interaction between charged particles.





Another functionals

$$I(u) = \Phi(\mu_{ks}^{(1,1)}, \dots, \mu_{ks}^{(i,j)}, \dots, \mu_{ks}^{(n,n)})$$

$$\mu_{ks}^{(i,j)} = \int_{M_{i,u}} (x_i - \overline{x}_i)^k (x_j - \overline{x}_j)^s \rho(t, x_i) dx_i$$

$$\overline{x}_i = \int_{M_{t,u}} x_i \rho(t, x_t) dx_t \qquad i, j = 1, \dots, n, \quad j \ge i$$

$$I(u) = \max_{t \in TN \in [0,T]} \max_{x \in M_{t,u}} \varphi(t, x, \rho(t, x)) \rightarrow \min_{u \in D}$$



MOM 4.

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(2)

(4)

$$\frac{dx}{dt} = f_1(t, x, u) + \int_{M_{t,u}} f_2(t, x, y_t) \rho(t, y_t) dy_t = f(t, x, u), x(0) = x_0 \in M_0$$
(1)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} f(t, x, u) + \rho \operatorname{div}_{x} f(t, x, u) = 0$$

$$\rho(0, x) = \rho_0(x), \rho = \rho(t, x)$$

$$\frac{dy}{dt} = h(t, x, y, u), \ y(0) = y_0$$
(3)

$$I(u) = \int_{0}^{T} \int_{M_{t,u}} \varphi(t, x_{t}, y(t, x_{t}), \rho(t, x_{t}), u) dx_{t} dt +$$

$$+ \int_{M_{T,u}} g(x_{T}, y(T, x_{T}), \rho(T, x_{T})) dx_{T} \to \min_{u \in D}$$



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$$\frac{dx}{dt} = f(t, x, u), \qquad (1$$

$$\frac{dy}{dt} = F_1(t, x, y, u) + \int_{M_{t,u}} F_2(t, x, y, z_t) \rho(t, z_t) dz_t = F(t, x, y, u),$$
(2)

$$\frac{d\rho}{dt} = -\rho(t, y) \cdot div_{y} F(t, x, y, u),$$
(3)

with initial conditions

$$\begin{aligned} x(0) &= x_0, \\ y(0) &= y_0 \in M_0, \\ \rho(0, y(0)) &= \rho_0(y_0), \quad y_0 \in M_0. \end{aligned}$$

Subsystem (1) describes dynamics of program (marked) motion.

Subsystem (2) describes dynamics of motions disturbed with respect to initial conditions. In particular, subsystem (2) can be considered as equations of deviations of program motions. Equation (3) is equation of change of charged distribution density of particles $\rho = \rho(t) = \rho(t, y(t))$ along trajectories (2) with given in initial moment $\rho_0(y_0)$ of distribution density of particles in set M_0 .

Vector-function F_1 determines influence of external fields on particle, and vector-function

 F_2 — interaction of particles. Let us note that only function F_1 depends on control u.





$$I(u) = I_{1}(u) + I_{2}(u), \quad I(u) \to \min_{u \in D}$$

$$I_{1}(u) = \int_{0}^{T} \varphi_{1}(t, x(t), u(t)) dt + g_{1}(x(T))$$
(5)

$$I_{2}(u) = \int_{0}^{\infty} \int_{M_{t,u}}^{\infty} \varphi_{2}(t, x(t), y_{t,u}(t)) dt + \int_{M_{T,u}}^{\infty} g_{2}(x(T), y_{t,u}) dy_{t}$$
(6)





Traveling Wave Accelerator

$$\frac{d\gamma}{d\xi} = -\alpha(\xi)\sin\varphi, \qquad (1)$$

$$\frac{d\varphi}{d\xi} = 2\pi \left(\frac{1}{\beta_{ph}(\xi)} - \frac{\gamma}{\sqrt{\gamma^2 - 1}}\right). \qquad (2)$$

 $lpha(\xi), \; eta_{\it ph}(\xi)$ - control functions





Radio-Frequency Quadrupole (RFQ): Modeling and Optimization.







Fig.2 Electric quadrupole



Fig.3 Modulation of RFQ electrodes to create longitudinal fields.





OPTIMIZATION CRITERIA

As the aim of RFQ structure optimization we consider following: ensurance of maximal capture of particles under the acceleration conditions; obtaining of required or maximal possible output energy.

If optimization is made only for these criteria, then final results can be obtained with high defocusing parameters and very small transverse acceptance. It means that restriction on defocusing parameter must be added

$$A_{def} = \frac{2k^2 \eta |\sin \varphi_s|}{(L/L_0)^2} = \frac{2k^2 u_1 |\sin u_2|}{x} \le A,$$
(1)

Parameter A usually lies in the range of A = 0.01 - 0.015.

If high-current beam is accelerated, then this beam must not be pinched in longitudinal direction in optimal mode. In this case new limitation that demands monotonic variation of rms beam width $\langle \Delta \varphi \rangle^2$. In ideal this limitation looks as

$$\frac{d\langle\Delta\varphi\rangle^2}{d\zeta} \le 0 \tag{2}$$







Phase portrait of beam in (ψ, ψ') plane and separatrix

Separatrix can be considered as the condition of particle capture into the acceleration mode.



OPTIMIZATION OF RADIAL MATCHING SECTION FOR RFQ CHANNEL

Ellipse matrices

$$G_x(t,\varphi_0)$$
, $G_y(t,\varphi_0)$

- t = 0 corresponds to the entrance into the radial matching section
- t = T corresponds to the entrance into the regular part of accelerator.

$$G_{x}(\mathbf{T},\varphi_{0}) = G_{x,\mathbf{T}}(\varphi_{0}), G_{y}(\mathbf{T},\varphi_{0}) = G_{y,\mathbf{T}}(\varphi_{0})$$
(1)

The optimization problem for the radial matching section is to find a function $a(\tau)$ i.e. law of the radius change along the matching sections, providing under the conditions (1) the maximum possible overlapping of families of ellipses at the entrance of the radial matching section.







Two approaches

$$\Phi_x(\varphi_0) = \operatorname{Sp}(G_x(0,\varphi_0) - B_x)^2$$
(2)

$$\Phi_{y}(\varphi_{0}) = \operatorname{Sp}(G_{y}(0,\varphi_{0}) - B_{y})^{2}$$
(3)

 B_x , B_y are given matrices Functional 1

$$U(a) = c_1 \int_{\varphi_1}^{\varphi_2} \Phi_x(\varphi_0) d\varphi_0 + c_2 \int_{\varphi_1}^{\varphi_2} \Phi_y(\varphi_0) d\varphi_0 \qquad (4)$$



Functional 2



$$\lambda_x^{-1}(\varphi_0) = \lambda^{-1}(G_x(0,\varphi_0),B_x)$$
(6)

$$\lambda_{y}^{-1}(\varphi_{0}) = \lambda^{-1}(G_{y}(0,\varphi_{0}),B_{y})$$
(7)

 $\lambda = \min(\lambda_1, \lambda_2)$ is a minimum eigenvalue of a cluster of quadratic forms generated by a pair of ellipses with the matrices *G* and *B*:

$$\chi(\lambda) = \det(G - \lambda B) = 0$$
, $\chi(\lambda_1) = \chi(\lambda_2) = 0$ (8)

The value of the inverse minimum eigenvalue characterizes the degree of mismatch pairs of ellipses. In the case of fully identical ellipses, this value is equal to unity. So always $\lambda^{-1} \ge 1$



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The analytic representations of the variations of the functionals (4),(5) were used to find geometric parameters of radial matching section of the RFQ accelerator of protons (initial energy 95keV, output energy 5 MeV, intervane voltage 100kV, RF field frequency 352 MHz, initial cell length 6.06 mm). Several of the possible choices of the law of variation of the channel radius along the radial matching section are presented in Figures 5, 8, 11. In Figures 1, 2 the RFQ acceptances without radial matching section are shown. The illustrations of effect of the radial matching sections (with channel radii presented at Figures 5, 8, 11) are shown in Figures 3-4, 6-7, 9-10 (correspondingly). The first variant (Figures 3-5) is rather usual and requires converging ellipses at the entrance of the radial matching section. But the two others give us unusual results with neutral and divergent input ellipses: Figures 6-8 and 9-11.





2



Fig.1 Acceptance without radial matching section, (X, dX/dz), Ex=0.050892π*sm*mrad

Fig.2 Acceptance without radial matching section, (Y, dY/dz), Ey=0.052122π*sm*mrad



Acceptance with radial matching section, (X, dX/dz), Ey= 0.18393π *sm*mrad

Acceptance with radial matching section, (Y, dY/dz), Ey= 0.16571π *sm*mrad



Fig. 5 Radius of channel in radial matching section



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Fig. 7



Acceptance with radial matching section, (X, dX/dz), Ex= 0.17945π *sm*mrad







Fig. 8 Radius of channel in radial matching section



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Acceptance with radial matching section, (X, dX/dz), Ey= $0.18023\pi^*$ sm*mrad





Fig. 10

Acceptance with radial matching section, (Y, dY/dz), Ey=0.16834π*sm*mrad

Fig. 11 Radius of channel in radial matching section





Self-consistence distributions

Uniform longitudinal magnetic field



The set Ω – admissibe values of integrals of transverse motion H and M.

Upper boundary: $H = \frac{M^2}{R^2} + \omega^2 R^2$. Lower boundary: $H = 2\omega |M|$. The *RV* distribution (represented by horizontal blue line)







A rigid rotator distribution (represented by blue line)



Another rigid rotator distribution (represented by blue line)







Linear combination of two rigid rotator distributions (represented by two blue lines)







Distribution with density $f(M, H) = \frac{\pi \rho_0}{2\omega^2 (M^2 - HR^2 + \omega^2 R^4)^{1/2}}$





Conclusion

Mathematical models for beam dynamics optimization were presented. They may be applied to different dynamical systems. The optimization approach to finding geometric parameters of radial matching section is considered. It should be noted, that the proposed approach can be applied to optimize the transverse dynamics in accelerators if the dynamics is described by linear or nonlinear equations. In the case of nonlinear equations it is needed to consider RMS characteristics of the beam. In particular, this method can be used to minimize the growth of the effective emittance in the accelerators.





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Acknowledgement

The author would like to thank deeply Oleg Drivotin, Alexander Ovsyannikov and Dmitry Ovsyannikov(Jr) for help in preparing presentation.





Thank you for your attention