VARIOUS MATRIX FORMALISM TO DESIGN ION LINEAR ACCELERATORS

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Abstract

The well-known betatron function parameterization in the beam optical computations provides an emittance independent representation of the properties of a beam transport system. The acceleration effects lead to nonsymplecticity of the transfer matrix. The error analysis of matrix presentation in the different phase spaces has been carried out. The coupling transformations of betatron functions for the studied phase spaces are presented.

INTRODUCTION

The traditional betatron functions $\beta, \alpha, \gamma = (1+\alpha^2)/\beta$ and μ (the betatron phase advance) are widely used to design the beam transport systems without acceleration [1]. They permit to decouple the problem of matching the characteristics of the injected beam to the acceptance of a transport system. However, the application of standard symplectic matrix mapping becomes unreliable when the acceleration effects are included. This effect depends from the choice of a phase space to design the beam transport system. Mainly this problem is important for the linear particle accelerators, where the accelerating parts are larger then parts without acceleration.

Formally, the single particle motion is described in a 6 – dimensional phase space. For simplicity we will restrict consideration to decoupled nondispersive beam transport systems, where the single particle dynamics for any direction is presented by 2×2 matrix. Therefore further only one transverse direction will be under study.

GENERAL FORMALISM

On the whole for any phase space the linear particle dynamics from longitudinal point s_1 to s_2 of a transport system may be described by matrix

$$M_{x,\varsigma} \equiv M_{x,\varsigma}(s_1, s_2) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}_{\varsigma} \quad , \qquad (1)$$

where x is the transverse coordinate and ς is the second phase space coordinate. In general the determinant of (1)

$$D_{\varsigma} = \det M_{x,\varsigma} = (m_{11}m_{22} - m_{12}m_{21})_{\varsigma}$$

(2)

may be not equal unity. In this case the transformation (1) will be nonsymplectic and beam emittance [1] will be not conserved. At position s in a transport line all beam particles lie within the ellipse [1]

$$\gamma^{\varsigma}(s)\,x^2 + 2\alpha^{\varsigma}(s)\,x\varsigma + \beta^{\varsigma}(s)\,\varsigma^2 = \varepsilon^{\varsigma}(s) \quad , \qquad (3)$$

where $\varepsilon^{\varsigma}(s)$ is the beam emittance.

According (1) the phase space particle dynamics between positions s_1 and s_2 is described by

$$\begin{pmatrix} x_2 \\ \varsigma_2 \end{pmatrix} \equiv \begin{pmatrix} x(s_2) \\ \varsigma(s_2) \end{pmatrix} = M_{x,\varsigma} \cdot \begin{pmatrix} x(s_1) \\ \varsigma(s_1) \end{pmatrix} \equiv M_{x,\varsigma} \cdot \begin{pmatrix} x_1 \\ \varsigma_1 \end{pmatrix} \quad .$$

(4)

Applying (3) for two mentioned above points and introducing the definitions

$$\varepsilon_1^{\varsigma} = \varepsilon^{\xi}(s_1)\ ,\ \varepsilon_2^{\varsigma} = \varepsilon^{\varsigma}(s_2)\ ,\ \varepsilon^{\varsigma}(s) = K_{\varsigma}(s)\cdot\varepsilon_1^{\varsigma} \quad , \qquad (5)$$

the following expressions for the betatron function propagation may be derived

$$\begin{pmatrix} \alpha_2^{\varsigma} \\ \beta_2^{\varsigma} \\ \gamma_2^{\varsigma} \end{pmatrix} = \frac{K_{\varsigma}}{D_{\varsigma}^2} \cdot T_{\varsigma} \cdot \begin{pmatrix} \alpha_1^{\varsigma} \\ \beta_1^{\varsigma} \\ \gamma_1^{\varsigma} \end{pmatrix} \quad , \qquad (6)$$

where $K_{\varsigma} \equiv K_{\varsigma}(s_2)$, and matrix T_{ς} has the standard form like for a beam transport system without acceleration [1]:

$$T_{\varsigma} = \begin{pmatrix} m_{11}m_{22} + m_{12}m_{21} & -m_{11}m_{21} & -m_{12}m_{22} \\ -2m_{11}m_{12} & m_{11}^2 & m_{12}^2 \\ -2m_{21}m_{22} & m_{21}^2 & m_{22}^2 \end{pmatrix}_{\varsigma} \quad .$$

From (6) it is possible to get the followed coupling coefficient for the betatron functions at different points of a beam transport system

$$\beta_2^{\varsigma}\gamma_2^{\varsigma} - (\alpha_2^{\varsigma})^2 = \frac{K_{\varsigma}^2}{D_{\varsigma}^2} \cdot \left(\beta_1^{\varsigma}\gamma_1^{\varsigma} - (\alpha_1^{\varsigma})^2\right) \quad . \qquad (7)$$

Whence it follows that for the arbitrary chosen phase space coordinates the next condition may be valid

$$F_{\varsigma}(s) \equiv \beta^{\varsigma}(s)\cdot\gamma^{\varsigma}(s) - \left(\alpha^{\varsigma}(s)\right)^2 \neq 1 \quad . \qquad (8)$$

CANONICAL PHASE SPACE

The canonical-conjugated variables are the coordinate and momentum (x, p) [1] for any transverse direction. For this phase space the beam emittance is an invariant [1]. Further the modified momentum is used:

$$p = \beta_x\gamma \quad ; \quad \beta_x = v_x/c \quad , \qquad (9)$$

where v_x is a transverse particle velocity; c is the light velocity; γ is a particle relativistic factor. In the formulas from previous section replacing $\varsigma \to p$ the following equalities are valid for the canonical phase space [1]:

$$D_p \equiv 1\ ;\ \varepsilon_1^p = \varepsilon_2^p\ ;\ K_p \equiv 1;\ F_p(s) \equiv 1 \quad . \qquad (10)$$

The matrix $M_{x,p}$ is symplectic and the beam emittance is invariant for the beam phase space motion [1]. Defining the betatron phase advance between positions s_1 and s_2 as

$$\Delta\mu_p = \mu(s_2) - \mu(s_1) \quad , \tag{11}$$

the elements of matrix $M_{x,p}$ may be calculated [2]:

$$m_{11}^p = \sqrt{\frac{\beta_2^p}{\beta_1^p}} \cdot \left(\cos\Delta\mu_p + \alpha_1^p \sin\Delta\mu_p\right)$$

$$m_{12}^p = \sqrt{\beta_1^p \beta_2^p} \cdot \sin\Delta\mu_p \tag{12}$$

$$m_{21}^p = -\frac{(1+\alpha_1^p\alpha_2^p)\sin\Delta\mu_p + (\alpha_2^p - \alpha_1^p)\cos\Delta\mu_p}{\sqrt{\beta_1^p\beta_2^p}}$$

$$m_{22}^p = \sqrt{\frac{\beta_1^p}{\beta_2^p}} \cdot \left(\cos\Delta\mu_p - \alpha_2^p \sin\Delta\mu_p\right) \quad .$$

And if the matrix elements are calculated by any way, the betatron function propagation (6) will be

$$\begin{pmatrix}\alpha_2^p \\ \beta_2^p \\ \gamma_2^p\end{pmatrix} = T_p \cdot \begin{pmatrix}\alpha_1^p \\ \beta_1^p \\ \gamma_1^p\end{pmatrix} \tag{13}$$

and $$\tan\Delta\mu_p = \frac{m_{12}^p}{m_{11}^p\beta_1^p - m_{12}^p\alpha_1^p} \quad . \tag{14}$$

The beam phase space ellipse (3) will be governed by

$$\gamma^p(s)\,x^2 + 2\alpha^p(s)\,xp + \beta^p(s)\,p^2 = \varepsilon^p(s) \equiv const \tag{15}$$

The above formalism is valid for Hamiltonian systems including both a beam transport system without acceleration and with acceleration [1]. However the exact matrix description in the canonical phase space (x, p) for the accelerating beam transport systems is difficult and has not a wide practice application.

NONCANONICAL PHASE SPACES

Standard Phase Space

The traditional representation of the beam transport systems is based on a noncanonical phase space (x, x'), where the angular divergence x' is determined as

$$x'(s) = p(s)/p_0(s) \quad , \tag{16}$$

here $p(s)$ is the modified canonical momentum (9) and $p_0(s)$ is the modified longitudinal momentum of the synchronous particle at a point of observation. Using (16) the coupling relations between the matrices $M_{x,p}$ and $M_{x,x'}$ are:

$$m_{11}^{x'} = m_{11}^p , \qquad m_{12}^{x'} = m_{12}^p p_0(s_1) ,$$

$$m_{21}^{x'} = m_{21}^p / p_0(s_2) , \; m_{22}^{x'} = m_{22}^p \, p_0(s_1)/p_0(s_2) \quad .$$

Taking into account (10) it follows that

$$D_{x'} = \frac{p_0(s_1)}{p_0(s_2)} \quad ; \qquad K_{x'} = \frac{p_0(s_1)}{p_0(s_2)} \quad ; \quad F_{x'}(s) \equiv const \quad . \tag{17}$$

From (17) it results that:

- matrix $M_{x,x'}$ is nonsymplectic and its determinant is reduced in an accelerating transport system, hence it does not admit a representation by (12)÷(14);
- the beam emittance $\varepsilon^{x'}(s)$ is "adiabatically damped" along a transport system with acceleration;
- the coupling coefficient (8) $F_{x'}(s)$ is the constant of motion.

If the matrix elements are calculated by any way, the betatron function propagation (6) will be

$$\begin{pmatrix}\alpha_2^{x'} \\ \beta_2^{x'} \\ \gamma_2^{x'}\end{pmatrix} = \frac{1}{D_{x'}} \cdot T_{x'} \cdot \begin{pmatrix}\alpha_1^{x'} \\ \beta_1^{x'} \\ \gamma_1^{x'}\end{pmatrix} \quad . \tag{18}$$

To calculate the betatron phase advance (11) the general definition [1] may be used

$$\Delta\mu_{x'} = \int_{s_1}^{s_2} \frac{ds}{\beta^{x'}(s)} \quad . \tag{19}$$

The beam phase space ellipse (3) in the phase space studied will be governed by

$$\gamma^{x'}(s)\,x^2 + 2\alpha^{x'}(s)\,xx' + \beta^{x'}(s)\,x'^2 = K_{x'}(s)\cdot\varepsilon^{x'}(s_1) \, . \tag{20}$$

Comparing (20) and (15) the following betatron function coupling for the phase spaces (x, x') and (x, p) is

$$\beta^{x'}(s) = \beta^p(s)\,p_0(s) \quad ; \quad \alpha^{x'}(s) = \alpha^p(s) \quad ; \tag{21}$$

$$\gamma^{x'}(s) = \gamma^p(s)/p_0(s) \quad ; \quad \varepsilon^{x'}(s) = \varepsilon^p/p_0(s) \quad .$$

From (10), (17) and (21) it follows that the coupling coefficient invariant is

$$F_{x'}(s) = F_p(s) \equiv 1 \quad . \tag{22}$$

Basing on the geometric characteristics of a phase ellipse [1], the real beam parameters at the fixed longitudinal point of a transport system are:

spot size $\quad x_{\max}(s) = \sqrt{\beta^{x'}(s)\varepsilon^{x'}(s)} \equiv \sqrt{\beta^p(s)\varepsilon^p}$;

divergence $\quad x'_{\max}(s) = \sqrt{\gamma^{x'}(s)\varepsilon^{x'}(s)} \equiv \sqrt{\gamma^p(s)\varepsilon^p}$.

Because of the betatron phase advance must be independent of the presentation chosen to describe the beam motion [2] the followed expression is valid

$$\Delta\mu_{x'} = \Delta\mu_p \quad . \tag{23}$$

The presented matrix formalism permits to simulate exactly the betatron function propagation in a beam transport system with acceleration if the matrix $M_{x,x'}$ was qualitatively determined. For example, the accelerating element models from [3] may be used.

Note, the equation (18) is coincided with the results of paper [2], but the algorithm presented in this paragraph is simpler and demands less number of the calculations.

Modified Phase Space

To study a beam transport system with acceleration the modified phase space (x, v), where $v \equiv v_x$ is the transverse particle velocity (9), was proposed. The independent variable is the time. Further the time markers t_1 and t_2 will be the moments when the synchronous particle traverses the longitudinal points s_1 and s_2.

Applying the previous section approach it follows:

$$D_v = \gamma(t_1)/\gamma(t_2)\ ,\ K_v = \gamma(t_1)/\gamma(t_2)\ ,\ F_v(t) \equiv const\ , \quad (24)$$

where $\gamma(t_1)$ and $\gamma(t_2)$ are the relativistic factors of the synchronous particle at the moments t_1 and t_2 respectively. From (24) it results that:

- matrix $M_{x,v}$ is nonsymplectic and its determinant is reduced in an accelerating transport system, hence it does not admit a representation by (12)÷(14);
- $D_v(t) > D_{x'}(s)$ and $K_v(t) > K_{x'}(s)$ for $t \leftrightarrow s$;
- the beam emittance $\varepsilon^v(t)$ is "adiabatically damped" along a transport system with acceleration;
- the coupling coefficient (8) $F_v(t)$ is the constant of motion.

The coupling relations between the matrices $M_{x,p}$ and $M_{x,v}$ are:

$$m_{11}^v = m_{11}^p\ , \qquad m_{12}^v = m_{12}^p\gamma(t_1)$$
$$m_{21}^v = m_{21}^p/\gamma(t_2)\ ,\ m_{22}^v = m_{22}^p\,\gamma(t_1)/\gamma(t_2) \quad .$$

The betatron function propagation is governed by

$$\begin{pmatrix} \alpha_2^v \\ \beta_2^v \\ \gamma_2^v \end{pmatrix} = \frac{1}{D_v}\cdot T_v \cdot \begin{pmatrix} \alpha_1^v \\ \beta_1^v \\ \gamma_1^v \end{pmatrix} \quad . \quad (25)$$

The betatron function parameterization (3) in the phase space (x,v) is

$$\gamma^v(t)\,x^2 + 2\alpha^v(t)\,xv + \beta^v(t)\,v^2 = K_v(t)\cdot\varepsilon^v(t_1) \quad . \quad (26)$$

The betatron function coupling for the phase spaces (x,v) and (x,p) is:

$$\beta^p(s) = \beta^v(t)\cdot c/\gamma(t) \quad ; \quad \alpha^p(s) = \alpha^v(t)$$
$$\gamma^p(s) = \gamma^v(t)\cdot\gamma(t)/c \quad ; \quad \varepsilon^v(t) = \varepsilon^p\cdot c/\gamma(t) \quad . \quad (27)$$

Therefore the coupling coefficient invariant is

$$F_v(t) = F_p(s) \equiv 1 \quad . \quad (28)$$

There is simple equality for the betatron phase advance (11) in the (x,v) and (x,x') phase spaces:

$$\Delta\mu_{x'} = \int_{s_1}^{s_2} \frac{ds}{\beta^{x'}(s)} = \int_{t_1}^{t_2} \frac{dt}{\beta^v(t)} = \Delta\mu_v \quad . \quad (29)$$

The real beam parameters at fixed time moment t corresponding to the longitudinal point s of a transport system are:

$$x_{\max}(t) = \sqrt{\beta^v(t)\varepsilon^v(t)} \equiv \sqrt{\beta^p(s)\varepsilon^p} = x_{\max}(s)\ ;$$
$$x'_{\max}(t) = \sqrt{\gamma^v(t)\varepsilon^v(t)} \equiv \sqrt{\gamma^p(s)\varepsilon^p} = x'_{\max}(s)\ .$$

PRACTICAL APPLICATION

Assuming the qualified presentation of matrices $M_{x,p}$, $M_{x,x'}$ and $M_{x,v}$, the algorithms presented above permit to get the reliable results of the desired betatron function propagation in a designed beam transport system even with the accelerating elements. In practice the accelerating elements are presented by the simplified models. For example, the proposed method [2] to propagate the betatron functions is complicated and used the "poor" rf-resonator model. In this case the calculation errors may be undesirable large. The widespread and more simple methods (for example [3]) exist to constrain a transfer matrix $M^*_{x,x'}$ or $M^*_{x,v}$ for a noncanonical phase space with the determinants

$$D^*_{x'} = 1 \text{ and } F^*_{x'}(s) = 1 \quad ; \quad D^*_v = 1 \text{ and } F^*_v(t) = 1 \quad . \quad (30)$$

The basic elements of a linear accelerator structure are the periodic parts with acceleration and transverse focusing (main parts), and matching sections for the different accelerator periodic parts. As a rule the beam energy gain both for the matching parts and main parts is insignificant compared with the beam energy. Therefore an application of the models with (30) may be effective in spite of the violations for some conditions (17) or (24).

On the base of the modified phase space the program complex was developed to design and simulate both the matching and periodic parts of the ion linear accelerators using the conditions (30). This complex was applied to match some periodic parts of the INR Linac [4]. In the Table 1 some parameters and determinants of the models are presented for two first matching parts of the INR Linac:

Table 1: Noncanonical Determinants

№	Input energy, MeV	Output energy, MeV	$D_{x'}$	D_v
1	100.1	113.3	0.937	0.987
2	139.3	158.6	0.933	0.982

Obviously to use for the simulation the modified phase space with (30) is more reliable from point of view to reduce the expected errors. For the periodic parts of the INR Linac the beam energy gain is ~3.7 MeV per period. It means that in the phase space (x,v) the beam transfer matrices are very close to condition (30), which leads to possibility to use the mathematical formalism (12)÷ (14).

CONCLUSIONS

The betatron function matrix formalism for the accelerating transport systems has been studied both for the canonical and noncanonical phase spaces. The exact formulas for the betatron function propagation are presented for all phase spaces studied. It was shown that the modeling of an ion accelerator in the phase space (x,v) does not create the additional problems for the matrix description of the standard elements of a beam transport system. Moreover for the low and medium beam energy ranges it permits to get a reliable solution for the betatron function propagation in an accelerating transport system by using the standard canonical matrix formalism.

REFERENCES

[1] E.Keil, “Single-particle dynamics – linear theory of perfect machines”, “Single-particle dynamics – linear machine lattices”, Theoretical aspects of the behaviour of beams in accelerators and storage rings, Geneva, p. 11 (1977).

[2] D.R.Douglas, J.Kewisch, R.C.York, “Betatron Function Parameterization of Beam Optics Including Acceleration”, LINAC’88, CEBAF-Report-89-001, Williamsburg, VA, USA, p. 328-331 (1988).

[3] K.R.Crandall, “Trace-3D documentation”, LA-11054-MS, USA (1987).

[4] S.E.Bragin et. al., “Transverse Beam Matching and Correction Procedures in INR LINAC”, LINAC’06, Knoxville, USA, p. 435 (2006); http://www.JACoW.org.