ANALYTICAL DESCRIPTION OF BETATRON OSCILLATIONS IN LINEAR MACHINES

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Abstract

As pointed out by Wiedemann, there are three basic periodic lattices using usually in storage rings [1,2]. Among these are the FODO, Chasman-Green lattice and triplet. They have not sextupole and octopole lenses and define so-called linear machines. To facilitate the calculations the consideration is restricted only to systems with quadrupole magnets. The equations of small deviations are derived in linear approximation. It is established that these expressions are specific differential equations with periodic coefficients and small parameter of the highest derivative. The Hill equations with a formula for the angular velocity determine the charged particle dynamics.

Given problem for storage rings is the continuation of electron dynamics study in accelerators [3,4] where the angular properties of synchrotron radiation were considered. There the equations are of Hill's type with small parameters was resolved by asymptotic methods. Finally, the features of synchrotron radiation were defined by means of β-function [4,5]. But a proposal about the use of the developed procedure to dynamics of particles in storage rings is revealed in this report. In addition, it was found that new modification of the Hill equation includes large parameter owing to the quadrupoles.

Let us take the FODO lattice, where symbols F and D mean focusing and defocusing quadrupoles, and O bending magnet. This magnetic system has N periods where one part respectively consists of focusing quadrupole of length a, free gap l long, bending dipole of length d, then through gaps defocusing quadrupole and bending magnet. It follows that one lattice has length

$L=2a+2d+4l$,

and magnitude of closed orbit

$S=2\pi R+(2a+4l)N$,

where R is the radius of bending magnet.

Let us accept that average radius can be found from relation

$$2\pi R_0 = 2\pi R + 2(a+2l)N,$$

where $2\pi R=2dN$. If put $k=(a+2l)/d$, then

$$R_0 = (1+k)R.$$

The magnetic field of dipole is $B_z = B$, and components of quadrupole define as

$$H_z^f = -gx,\ \ H_z^d = gx,\ \ H_x^f = -gz,\ H_x^d = gz,$$

where g is the lens constant, index f and d means focusing and defocusing. Coordinate x coincides with a radial direction.

The vertical components of magnetic field depend on azimuth angle φ as

$$-gx,\quad \varphi\in[0, aT^n];$$
$$B,\ \ \varphi\in[(a+l)T^n, (a+l+d)T^n];$$
$$gx,\quad \varphi\in[\pi/N, (2a+2l+d)T^n];$$
$$B, \varphi\in[(2a+3l+d)T^n, (2a+3l+2d)T^n],$$

where $T^n = 2\pi/(NL)$.

After a Fourier series expansion this magnetic field takes the form

$$H_z = \frac{2d}{L}B + \sum_{k=1}^{\infty}[\frac{4B}{\pi}\frac{(-1)^k}{k}\cos\frac{\pi k}{2}\sin\frac{\pi k}{L}d -$$
$$\frac{2}{\pi}gx\frac{1-(-1)^k}{k}\sin k\tau_1]\cos k(\tau-\tau_1),$$

where $\tau = N\varphi,\ \ \tau_1 = \pi a/L$. In particular, in studies of a radiation problem the dipole magnetic field can be averaged and

$$\bar{H}_z = \frac{2d}{L}B - gxf(\tau), \quad (1)$$

where $f(\tau) = (4/\pi)n(\tau)$,

$$n(\tau) = \sum_{\nu=0}^{\infty}\frac{\sin(2\nu+1)\tau_1}{2\nu+1}\cos(2\nu+1)(\tau-\tau_1).$$

By analogy, one may derive $H_r = -gzf(\tau)$.

Besides an angular velocity is

$$\dot{\varphi} = \frac{\omega_0}{1+k}(1-\frac{x}{R_0}+\frac{3}{2}\frac{x^2}{R_0^2}) +$$
$$\frac{\omega_q}{R_0^2}\int f(\tau)(z\dot{z}-x\dot{x})dt, \quad (2)$$

where

$$\omega_0 = \frac{e_0B}{m_0c},\ \ \omega_g = \frac{e_0gR_0}{m_0c},\ \ \frac{2d}{L} = \frac{1}{1+k}.$$

Equations of betatron oscillations become

$$\frac{d^2z}{d\tau^2} + \frac{1}{N^2}\frac{(1+k)\omega_g}{\omega_0}f(\tau)z = 0, \quad (3)$$

$$\frac{d^2x}{d\tau^2}+\frac{1}{N^2}[1-\frac{(1+k)\omega_g}{\omega_0}f(\tau)]x=0. \quad (4)$$

Let us introduce new constants

$$C=\frac{gR_0(1+k)}{B},\ \lambda^2=\frac{4C}{\pi N^2}.$$

Under prevailing g and B it turns out that parameter $\lambda>>1$. Eq. (3) can be rewritten as

$$\frac{d^2z}{d\tau^2}+\lambda^2 n(\tau)z=0. \quad (5)$$

Expression (5) is the Hill equation with a large parameter. Besides it is differential equation with periodic coefficient having small parameter at higher derivative. In literature such a type of equations is absent. To seek its solution one may test the WKB-method. Then if put

$$z=\exp(i\lambda\int\sqrt{n(\tau)}d\tau)\cdot\varphi(\tau),$$

terms with λ^2 in (5) are cancelled and in the first approximation on power $1/\lambda$ function

$$\varphi(\tau)=1/\sqrt[4]{n(\tau)}.$$

But applicable asymptotics of Eq.(5) cannot depend on $n(\tau)$ because it is differentiated two times and leads to the divergent series.

If one takes new variable

$u=\lambda^2\tau$ expression (5) can be represented as

$$\frac{d^2z}{du^2}+\frac{1}{\lambda^2}n(\frac{u}{\lambda^2})z=0,$$

and may be resolved as equation with small parameter. Suppose that a solution has the form

$$z=\exp(i\gamma u)\cdot\varphi(u),$$

where γ and $\varphi(u)$ are developed in a power series.

Here the frequency of dominant oscillations is proportional to

$$\frac{\pi^2 a}{4L}\sqrt{1-\frac{4a}{3L}},$$

but asymptotics will be increase.

Let us pass on the Chasman-Green lattice. In this case there are in centre focusing quadrupole a_1 long , then to both side disperce straight sections of length l_1,bending magnets of length d, free gaps l long, defocusing quadrupoles a long, gaps l long, focusing quarupoles of length a and gaps l_2 long (see in [2] Fig.13.2).

For one period L is

$$2d+4a+a_1+4l+2l_1+2l_2.$$

Corresponding equation for vertical oscillations may be expressed in the form

$$\frac{d^2z}{d\tau^2}+\frac{C}{N^2}(\frac{a_1}{L}+\frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{f_1}{\nu}\cos\nu\tau)z=0, \quad (6)$$

where $k=(L-2d)/2d$,

$$f_1=4\sin\tau_2 a\sin\tau_2(a+l)\sin\tau_2(2a+l+2l_2)+$$

$$(-1)^\nu\sin\tau_2 a_1,\ \tau_2=\pi\nu/L.$$

Contrary to the Eq.(5) given relation contains additional constant part along with the trigonometric series.

An equation for triplet achromat lattice in form is close to Eq.(6). Here there is in centre defocusing quadrupoleof length a_1, then laterally lies through free drifts focusing quadrupole of a long and bending magnet of d long. For example, path of right side is equal to

$$a_1+l_1+a+l_2+d+l_3,$$

where l_i is the length of straight sections.

In linear case the equation of axial oscillations becomes

$$\frac{d^2z}{d\tau^2}+\frac{C}{N^2}(\frac{2a-a_1}{L}+\frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu}f_2\cos\nu\tau)z=0, \quad (7)$$

where $C=\omega_q(1+k)/\omega_0$,

$$f_2=2\sin\tau_2 a\cos\tau_2(2l_1+a+a_1)-\sin\tau_2 a_1.$$

Formulas (4)-(7) are the new differential equations. Conditions of formulated theorems in available literature [6,7] differ from ones of given paper. In particulare , one cannot differentiate periodic coefficients because will be broken convergence of series. The Hill method [8] also cannot be used since the infinite determination is increased at $\lambda>>1$.

Moreover, there are other decision methods of relations containing a large parameter, for example, transformed Airy equation, differential equations with turning points but in these cases periodic coefficients are absent.

Nevertheless, the derived equations permit to carry out the simulation. Taking into account the injection of particles it is possible to introduce the initial conditions and resolve the Cauchy problem. Moreover, the analyzed examples one may in the framework of this procedure go over to more complex structures adding the devices with their transverse components of magnetic field.

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