CORRECTION TERMS TO PANOFSKY-WENZEL FORMULA AND WAKE POTENTIAL

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Abstract

In their 1956 article [1] Panofsky and Wenzel derived a relation for the net transverse kick experienced by a fast charge particle crossing a closed cavity excited in a single rf mode. Later this relation, usually referred to the Panofsky-Wenzel theorem, was generalized for cavity containing wake field induced by a driving charge. This theorem has played very important role in accelerator physics. One well-known conclusion of this paper was that in a TE mode the deflecting impulse of the electric field always cancels the impulse of the magnetic fields. In our presentation we more exactly rederive Panofsky and Wenzel's result and obtain correction terms to the transverse kick. We show that in a TE mode the net transverse kick is not zero. Using the given approach we find correction terms to wake potentials which turn out to be inversely proportional to the relativistic factor. Practical implications of our results are discussed.

INTRODUCTION

The well-known Panofsky-Wenzel formula [1] is concerned with the net transverse kick experienced by a fast charged particle crossing a closed cavity containing rf fields

$$\Delta\vec{p}_{\perp} = e\int_0^L \vec{\nabla}_{\perp} A_z\Big|_{t=z/\mathrm{v}_z} dz \cdot \tag{1}$$

It is the practical tool in dynamic of ultrarelativistic beams interacting with rf structures. In Eq.(1) e is the charge of particle, v_z is the longitudinal velocity close to the speed of light c, L is the length of cavity, $\vec{\nabla}_{\perp}A_z$ is the transverse gradient of the longitudinal component of rf vector potential. In the wake potential theory the relation (1), usually referred to as the Panofsky-Wenzel theorem, was generalized for rf cavities and infinitely repeating structures containing wake field induced by a driving charge [2]. Some reformulated versions of the Panofsky-Wenzel theorem are given in Ref. [3] for study of rf asymmetry in photo-injectors, in Ref. [4] for the case in which phase slippage between the wave and beam is not negligible. The interesting interpretation of paper [1] results can be found in [5].

One well-known conclusion, that in a TE mode the deflecting impulse of the electric field always cancels the impulse of the magnetic field, follows from Eq.(1). However, generally, if A_z is zero or small enough, the formula (1) is not true. The fact is that the Panofsky-Wenzel theorem assumes in its derivation that the particle experiencing Lorentz force moves parallel to the z-axis at constant velocity $\vec{\mathrm{v}} = \vec{\mathrm{v}}_z + \vec{\mathrm{v}}_{\perp} \approx \vec{\mathrm{v}}_z$. So, in the dot product $\vec{\mathrm{v}}\vec{A} = \mathrm{v}_z A_z + \vec{\mathrm{v}}_{\perp}\vec{A}_{\perp}$ the second term was neglected. In the case of A_z=0 or $\mathrm{v}_z A_z \ll \vec{\mathrm{v}}_{\perp}\vec{A}_{\perp}$, it is necessary to take into account the transverse momentum imparted to the particle during its transit time through the cavity. In this paper we will derive more exactly the Panofsky and Wenzel's relation and obtain correction terms to it. As well we will discuss possibility to measure phase volume of a bunch with rf deflector based on a TE mode. Finally we will attempt to find correction terms to the wake potential.

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REDERIVING THE THEOREM

Following to the Panofsky-Wenzel derivation [1], the equation of motion of the particle in terms of a vector potential is given

$$\frac{d\vec{p}}{dz} = \frac{e}{\mathrm{v}_z}\left[-\frac{\partial\vec{A}}{\partial t} + \vec{\mathrm{v}}\times\vec{\nabla}\times\vec{A}\right]\Bigg|_{t=z/\mathrm{v}_z}, \tag{2}$$

where $dz=\mathrm{v}_z dt$. Using the following expressions $\vec{\mathrm{v}}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}\left(\vec{\mathrm{v}}\vec{A}\right) - \left(\vec{\mathrm{v}}\vec{\nabla}\right)\vec{A}$, $\frac{\partial\vec{A}}{\partial t} = \frac{d\vec{A}}{dt} - \left(\vec{\mathrm{v}}\vec{\nabla}\right)\vec{A}$, and expressing the particle velocity as $\vec{\mathrm{v}} = \vec{\mathrm{v}}_z + \mathrm{v}_z\vec{p}_{\perp}/p_z$, (where $\vec{p}_{\perp}$ and p_z are the transverse and longitudinal momentums, respectively) we can write the equation for transverse momentum as

$$\frac{d\vec{p}_{\perp}}{dz} = e\left\{-\frac{d\vec{A}_{\perp}}{dz} + \vec{\nabla}_{\perp}\left(A_z + \frac{\vec{p}_{\perp}}{p_z}\vec{A}_{\perp}\right)\right\}\Bigg|_{t=z/\mathrm{v}_z}. \tag{3}$$

Integrating Eq.(3) we obtain the dependence of the transverse momentum on a coordinate z

$$\begin{aligned}\vec{p}_{\perp}(z) &= \vec{p}_{0,\perp} - e\vec{A}_{\perp}\left(\vec{r}_{\perp}, z, t\right)\Big|_{t=z/\mathrm{v}_z} \\ &+ e\int_0^z \vec{\nabla}_{\perp}\left(A_z\left(\vec{r}_{\perp}, z, t\right) + \frac{\vec{p}_{\perp}}{p_z}\vec{A}_{\perp}\left(\vec{r}_{\perp}, z, t\right)\right)\Bigg|_{t=z/\mathrm{v}_z} dz\end{aligned}, \tag{4}$$

where it is assumed that $\vec{A}_{\perp} = 0$ at z=0 and z=L (the cavity end walls are normal the z-diraction or the path of the particle begins and ends in a field-free region), $\vec{p}_{0,\perp}$ is the initial transverse momentum, $\vec{r}_{\perp}$ is the transverse coordinate of the charge. Due to the small parameter $p_{\perp}/p_z \ll 1$, the integral equation Eq.(4) may be solved by the successive approximations. Therefore we expand it into series on the small parameter

$$\begin{aligned}\vec{p}_{\perp} &= \vec{p}_{0,\perp} - e\left[1 + \delta\vec{r}_{\perp}\cdot\vec{\nabla}_{\perp}\right]\vec{A}_{\perp}\left(\vec{r}_{0\perp}, z, t\right)\Big|_{t=z/\mathrm{v}_z} \\ &+ e\int_0^z \vec{\nabla}_{\perp}\left(\left[1 + \delta\vec{r}_{\perp}\cdot\vec{\nabla}_{\perp}\right]A_z\left(\vec{r}_{0\perp}, z, t\right) + \frac{\vec{p}_{\perp}}{p_z}\vec{A}_{\perp}\left(\vec{r}_{0\perp}, z, t\right)\right)\Bigg|_{t=z/\mathrm{v}_z} dz \\ &+ o\left(p_{\perp}^2/p_z^2\right).\end{aligned} \tag{5}$$

Here $\delta\vec{r}_\perp = \int_0^z (\vec{p}_\perp / p_z)\, dz$ and it is assumed that $\left|(\delta\vec{r}_\perp \cdot \vec{\nabla}_\perp)\vec{A}\right| \ll |\vec{A}|$, $\vec{r}_{0,\perp}$ is the initial transverse coordinate of the charge. Firstly from Eq.(5) we find the zero order approximation of the transverse momentum as function of z-coordinate

$$\begin{aligned}\vec{p}_\perp^{(0)}(z) &= \vec{p}_{0,\perp} - e\vec{A}_\perp(\vec{r}_{0\perp}, z, t)\Big|_{t=z/\mathrm{v}_z} \\ &+ e\int_0^z \vec{\nabla}_\perp A_z(\vec{r}_{0\perp}, z, t)\Big|_{t=z/\mathrm{v}_z} dz\end{aligned} . \qquad (6)$$

We see that at $z=L$ the zero order approximation Eq.(6) reduces to the Panofsky-Wenzel formula (1). Substituting Eq.(6) into Eq.(5) we obtain the transverse momentum imparted to the particle with the accuracy of the first order approximation

$$\begin{aligned}\Delta\vec{p}_\perp &= e\int_0^L \vec{\nabla}_\perp A_z dz + e\int_0^L \vec{\nabla}_\perp\left(\int_0^z \frac{\vec{p}_{0,\perp}}{p_z} dz\right)\cdot\vec{\nabla}_\perp A_z dz \\ &- e^2\int_0^L \vec{\nabla}_\perp\left(\int_0^z \frac{\vec{A}_\perp}{p_z} dz' - \int_0^z \frac{dz'}{p_z}\int_0^{z'} \vec{\nabla}_\perp A_z dz''\right)\cdot\vec{\nabla}_\perp A_z dz \\ &+ \int_0^L \vec{\nabla}_\perp\left(\frac{\vec{p}_{0,\perp} e\vec{A}_\perp}{p_z} - \frac{(eA_\perp)^2}{p_z} + \frac{e^2\vec{A}_\perp}{p_z}\int_0^z \vec{\nabla}_\perp A_z dz'\right) dz\end{aligned} , \qquad (7)$$

where $\vec{A} \equiv \vec{A}(\vec{r}_{0\perp}, z, t)\Big|_{t=z/\mathrm{v}_z}$.

From the Eq.(7) we see that in the case of exciting a TE mode $\vec{A}_z = 0$ the net transverse kick is

$$\Delta\vec{p}_\perp = \int_0^L \vec{\nabla}_\perp\left(\frac{\vec{p}_{0,\perp}}{p_z} e\vec{A}_\perp - \frac{|eA_\perp|^2}{p_z}\right)\Bigg|_{t=z/\mathrm{v}_z} dz \, . \qquad (8)$$

As seen from the Eq.(8), even if $\vec{p}_{0,\perp} = 0$ the ponderomotive force, which is square on the transverse component of vector potential, ensures the non-zero transverse momentum imparted to the particle.

CONCEPT OF MEASUREMENT OF PHASE VOLUME BY TE MODE DEFLECTOR

Eq.(8) shows that the transverse momentum imparted to the particle by a TE mode dependences on the initial transverse momentum. That may point to ability to measure phase volume of a beam by using a TE mode deflector. Let us rewrite Eq.(8) in components as

$$\begin{aligned}\mathrm{v}_{0,x} a_{xx} + \mathrm{v}_{0,y} a_{xy} &= f_x, \\ \mathrm{v}_{0,x} a_{yx} + \mathrm{v}_{0,y} a_{yy} &= f_y,\end{aligned} \qquad (9)$$

where

$$a_{xx} = e\int_0^L \frac{\partial}{\partial x} A_x\Big|_{t=z/\mathrm{v}_z} \frac{dz}{\mathrm{v}_z}, \quad a_{xy} = e\int_0^L \frac{\partial}{\partial x} A_y\Big|_{t=z/\mathrm{v}_z} \frac{dz}{\mathrm{v}_z},$$

$$a_{yx} = e\int_0^L \frac{\partial}{\partial y} A_x\Big|_{t=z/\mathrm{v}_z} \frac{dz}{\mathrm{v}_z}, \quad a_{yy} = e\int_0^L \frac{\partial}{\partial y} A_y\Big|_{t=z/\mathrm{v}_z} \frac{dz}{\mathrm{v}_z},$$

$$\begin{aligned}f_x &= e^2\int_0^L \frac{\partial A_\perp^2}{\partial x}\Big|_{t=z/\mathrm{v}_z} \frac{dz}{p_z} - \Delta p_x, \\ f_y &= e^2\int_0^L \frac{\partial A_\perp^2}{\partial y}\Big|_{t=z/\mathrm{v}_z} \frac{dz}{p_z} - \Delta p_y.\end{aligned} \qquad (10)$$

For the case

$$a_{xx} = 0, \qquad a_{yy} = 0. \qquad (11)$$

The solution of the equation set (9) is

$$\mathrm{v}_{0,x} = \frac{f_y}{a_{yx}}, \quad \mathrm{v}_{0,y} = \frac{f_x}{a_{xy}}. \qquad (12)$$

An ultrarelativistic beam

For a case of ultrarelativictic particles, ($\mathrm{v}_z=c$, $\gamma\to\infty$, where γ is Lorentz factor) Eqs.(12) can be simplified

$$\mathrm{v}_{0,x} = -\frac{m_0c^2\gamma}{e\int_0^L \frac{\partial}{\partial y} A_x\Big|_{\substack{t=z/c\\ y=y_0}} dz}\frac{\Delta y}{l}, \; \mathrm{v}_{0,y} = -\frac{m_0c^2\gamma}{e\int_0^L \frac{\partial}{\partial x} A_y\Big|_{\substack{t=z/c\\ x=x_0}} dz}\frac{\Delta x}{l}. \qquad (13)$$

Here the transverse kick (Δp_x, Δp_y) is expressed through a beam deflecting from axis ($\Delta x= x-x_0$, $\Delta y=y-y_0$) in a drift tube of length l which is stationed after the cavity, $\Delta p_x=m_0c\gamma\Delta x/l$ $\Delta p_y=m_0c\gamma\Delta y/l$, m_0 is the rest mass, (x_0, y_0) and (x, y) are the transverse coordinates of a particle at the entry of the cavity and the drift tube exit, correspondently.

Let us consider a rectangular box where the following TE modes are excited

$$A_x = -B_0\frac{b}{n}\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{L}z\right)\cos\left(\omega_{n,p}t+\varphi\right), \quad A_y = 0 \qquad (14)$$

at the eigenfrequency $\omega_{n,p} = \pi c\sqrt{(n/b)^2 + (p/L)^2}$, and

$$A_x = 0, \; A_y = B_0\frac{a}{m}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{p\pi}{L}z\right)\cos\left(\omega_{m,p}t+\varphi\right) \qquad (15)$$

with the eigenfrequency $\omega_{m,p} = \pi c\sqrt{(m/a)^2 + (p/L)^2}$, where a, b, L are the edge lengths of the rectangular box in x-, y -, and z - directions, correspondently, B_0 is the constant, φ is the initial phase.

We write

$$\begin{aligned}&x_0 = \langle x_0\rangle + \delta x_0, \; y_0 = \langle y_0\rangle + \delta y_0, \varphi = \varphi_c + \delta\varphi \\ &x = \langle x\rangle + \delta x, \; y = \langle y\rangle + \delta y,\end{aligned} \qquad (16)$$

where <…> is the operator of averaging over particles, φ_c is the reference phase.

We assume that the bunch to be short $|\delta\varphi| << 2\pi$. Setting $\langle x_0\rangle=a/2$, $\langle y_0\rangle=b/2$, and substituting Eq.(14, 15) for even n and m into Eq. (13) we obtain

$$\begin{aligned}\gamma\frac{\Delta y}{l} &= \beta_{0,x}(-1)^{\frac{n}{2}+1}\frac{eB_0}{m_0c}\frac{p}{L}\left(\frac{b}{n}\right)^2\left\{\cos\varphi_c - (-1)^p\cos\left(\frac{\omega_{n,p}L}{c}+\varphi_c\right)\right. \\ &\left. -\delta\varphi\left[\sin\varphi_c - (-1)^p\sin\left(\frac{\omega_{n,p}L}{c}+\varphi_c\right)\right]\right\},\end{aligned} \qquad (17)$$

$$\gamma\frac{\Delta x}{l}=\beta_{0,y}\left(-1\right)^{\frac{m}{2}}\frac{eB_0}{m_0c}\frac{p}{L}\left(\frac{a}{m}\right)^2\left\{\cos\varphi_c-\left(-1\right)^p\cos\left(\frac{\omega_{m,p}L}{c}+\varphi_c\right)\right.$$
$$\left.-\delta\varphi\left[\sin\varphi_c-\left(-1\right)^p\sin\left(\frac{\omega_{m,p}L}{c}+\varphi_c\right)\right]\right\}. \quad (18)$$

Setting the reference phase $\varphi_{c,k}$ (where k=n,m) at which

$$\sin\varphi_{c,k}-\left(-1\right)^p\sin\left(\frac{\omega_{k,p}L}{c}+\varphi_{c,k}\right)=0, \quad (19)$$

we can obtain the initial transverse characteristic bunch

$$\left\langle\beta_{0,x}^2\right\rangle=\left\langle\gamma^2\right\rangle\frac{\left\langle y^2\right\rangle-\left\langle y_0{}^2\right\rangle-2\left\langle y_0\right\rangle\left(\left\langle y\right\rangle-\left\langle y_0\right\rangle\right)}{\left(2\frac{eB_0}{m_0c}\frac{p}{L}\left(\frac{b}{n}\right)^2\right)^2l^2},$$
$$\left\langle\beta_{0,y}^2\right\rangle=\left\langle\gamma^2\right\rangle\frac{\left\langle x^2\right\rangle-\left\langle x_0{}^2\right\rangle-2\left\langle x_0\right\rangle\left(\left\langle x\right\rangle-\left\langle x_0\right\rangle\right)}{\left(2\frac{eB_0}{m_0c}\frac{p}{L}\left(\frac{a}{m}\right)^2\right)^2l^2}, \quad (20)$$

$$\left\langle\beta_{0,x}\right\rangle=\left\langle\gamma\right\rangle\frac{\left\langle y\right\rangle-\left\langle y_0\right\rangle}{\left(-1\right)^{\frac{n}{2}+1}2\frac{eB_0}{m_0c}\frac{p}{L}\left(\frac{b}{n}\right)^2l},$$
$$\left\langle\beta_{0,y}\right\rangle=\left\langle\gamma\right\rangle\frac{\left\langle x\right\rangle-\left\langle x_0\right\rangle}{\left(-1\right)^{\frac{m}{2}}2\frac{eB_0}{m_0c}\frac{p}{L}\left(\frac{a}{m}\right)^2l}. \quad (21)$$

Assuming $<\delta\varphi>=0$ and setting $\varphi'_{c,k}$ (where k=n,m) at which

$$\cos\varphi'_{c,k}-\left(-1\right)^p\cos\left(\frac{\omega_{k,p}L}{c}+\varphi'_{c,k}\right)=0, \quad (22)$$

we can obtain the longitudinal characteristic bunch

$$\left\langle\delta\varphi^2\right\rangle=\left\langle\gamma^2\right\rangle\frac{\left\langle y'^2\right\rangle-\left\langle y_0{}^2\right\rangle}{\left(2\frac{eB_0}{m_0c}\frac{p}{L}\left(\frac{b}{n}\right)^2\right)^2l^2\left\langle\beta_{0,x}^2\right\rangle},$$
$$\left\langle\delta\varphi^2\right\rangle=\left\langle\gamma^2\right\rangle\frac{\left\langle x'^2\right\rangle-\left\langle x_0{}^2\right\rangle}{\left(2\frac{eB_0}{m_0c}\frac{p}{L}\left(\frac{a}{m}\right)^2\right)^2l^2\left\langle\beta_{0,y}^2\right\rangle}. \quad (23)$$

CORRECTION TERMS TO WAKE POTENTIAL

Using the approach developed above we consider wake fields ($\vec{E},\vec{B}$) in terms of vector and scalar potentials $\vec{A},\Phi$

$$\vec{E}=-\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\Phi,\ \vec{B}=\vec{\nabla}\times\vec{A},\ \vec{\nabla}\vec{A}=0 \quad (24)$$

excited by point charge q traversing the cavity at velocity $\vec{\mathrm{v}}=\vec{\mathrm{v}}_z+\vec{\mathrm{v}}_\perp,\ \ \mathrm{v}_z\approx c$. Let a test charge e follows with the same velocity at distance s from the exciting point-charge q. The equation for the kick experienced by the test particle in the wake field may be given

$$\frac{d\vec{p}}{dz}=e\left\{-\frac{d\vec{A}}{dz}+\vec{\nabla}\left(A_z-\frac{\Phi}{\mathrm{v}_z}+\frac{\vec{p}_\perp}{p_z}\vec{A}_\perp\right)\right\}\Bigg|_{t=(s+z)/\mathrm{v}_z}. \quad (25)$$

Integrating Eq.(25) over (0, z), then, expanding $\vec{p}$ into series on the small parameter $p_\perp/p_z \ll 1$, we find the zero order transverse momentum as function of z

$$\vec{p}_\perp^{(0)}\left(z,s\right)=\vec{p}_{0,\perp}-e\vec{A}_\perp\left(z,s\right)+e\int_0^z\vec{\nabla}_\perp\left(A_z-\frac{\Phi}{\mathrm{v}_z}\right)\Bigg|_{t=(s+z)/\mathrm{v}_z}dz. \quad (26)$$

Further for simplicity we assume that the path of the particle begins and ends in a field-free region, $\vec{A}(z=0)=\vec{A}(z=L)=0$. Substituting Eq.(26) into Eq.(25), and taking into account the definition [2], we obtain the wake potential with the correction terms

$$\vec{W}\left(s\right)\equiv\frac{\mathrm{v}_z\Delta\vec{p}\left(s\right)}{eq}=\frac{1}{q}\int_0^L\left[\vec{\nabla}\left(\mathrm{v}_zA_z-\Phi+u\right)\right]\Big|_{t=(s+z)/\mathrm{v}_z}dz, \quad (27)$$

where u associates the correction terms

$$u\equiv\int_0^z\frac{\vec{p}_\perp^{(0)}}{p_z}dz\cdot\vec{\nabla}_\perp\left(\mathrm{v}_zA_z-\Phi\right)+\mathrm{v}_z\frac{\vec{p}_\perp^{(0)}}{p_z}\vec{A}_\perp\Bigg|_{t=(s+z)/\mathrm{v}_z}. \quad (28)$$

Substituting Eq.(26) into Eq.(28) we obtain

$$u=\int_0^z\left(\frac{\vec{p}_{0,\perp}}{p_z}-\frac{e\vec{A}_\perp}{p_z}\right)dz\cdot\vec{\nabla}_\perp\left(\mathrm{v}_zA_z-\Phi\right)$$
$$+e\int_0^z\left(\frac{dz'}{p_z\mathrm{v}_z}\int_0^{z'}\vec{\nabla}_\perp\left(\mathrm{v}_zA_z-\Phi\right)dz''\right)\cdot\vec{\nabla}_\perp\left(\mathrm{v}_zA_z-\Phi\right) \quad (29)$$
$$+\mathrm{v}_z\frac{\vec{p}_{0,\perp}\vec{A}_\perp}{p_z}-\mathrm{v}_z\frac{eA_\perp^2}{p_z}+\frac{e\vec{A}_\perp}{p_z}\int_0^z\vec{\nabla}_\perp\left(\mathrm{v}_zA_z-\Phi\right)dz.$$

As seen from the Eq.(29) the correction terms to the wake potential are proportional to γ^{-1}, whereas the modern wake theory [2] gives the correction terms which are proportional to γ^{-2}.

SUMMARY

Rederiving the Panofsky and Wenzel's theorem we obtained the correction terms to the net transverse kick which is not zero in the TE mode. That allows to use the TE mode deflector to measure phase volume of a beam. We found the correction terms to wake potentials which are shown to be inversely proportional to the relativistic factor.

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