

# NONLINEAR CHARACTERISTICS OF THE TME CELL

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## Abstract

The TME (Theoretical Minimum Emittance) cell is being used now for designing the lattice of different storage rings (SR sources, damping rings, FFAG accelerators, etc.). Strong sextupoles required to correct the natural chromaticity of the lattice reduce the dynamic aperture. In the paper we consider the main features of the nonlinear perturbation strength and its connection with the essential lattice parameters: horizontal emittance, betatron tunes, and natural chromaticity. The analytical results are compared with the computer simulation.

## INTRODUCTION

Each author should submit all of the source files (text and figures), the postscript file and a hard copy version of the paper. This will allow the editors to reconstruct the paper in case of processing difficulties and compare the version produced for publication with the hard copy. The TME lattice [1] has recently become popular in designing accelerators for different applications [2,3,4]. Linear optical properties of this lattice have been investigated well and can be found elsewhere. In particular, a thorough and detailed report [5] describing application of the TME cell for linear collider damping ring design is worth mentioning.

Below we study features of nonlinear motion of a particle in the TME lattice, especially the relation between the strength of nonlinear perturbation and such characteristics of circular accelerator as horizontal emittance, natural chromaticity, betatron tunes, etc. Chromatic sextupoles are assumed to be the main source of nonlinear perturbation and only transverse motion is considered.

The main (resonant) azimuthal harmonics are used below for estimation of the sextupole perturbation strength and its relations with essential accelerator parameters. Earlier, a similar approach was used in [6] to study the DBA synchrotron light source lattice.

## RESONANT HARMONICS

An azimuthal Fourier expansion of the sextupole Hamiltonian for single particle motion has the form

$$\begin{aligned}
 H = & \nu_x J_x + \nu_y J_y \\
 & + (2J_x)^{3/2} \sum_n [3A_{1n} \cos(\varphi_x - n\theta) + A_{3n} \cos(3\varphi_x - n\theta)] \\
 & - 3(2J_x)^{1/2} (2J_y) \sum_n [2B_{1n} \cos(\varphi_x - n\theta) + B_{+n} \cos(\varphi_+ - n\theta) \\
 & + B_{-n} \cos(\varphi_- - n\theta)], \quad (1)
 \end{aligned}$$

where  $\theta = s/R$  is the azimuthal angle (an independent variable),  $R$  is the average orbit radius and the five types of harmonics ( $j = 1, 3$ )

$$\begin{aligned}
 A_{jn} &= \frac{1}{48\pi} \sum_m \beta_{xm}^{3/2} (k_2 l)_m \cos(j\psi_x - \nu\theta + n\theta)_m, \\
 B_{1n} &= \frac{1}{48\pi} \sum_m \beta_{xm}^{1/2} \beta_{ym} (k_2 l)_m \cos(\psi_x - \nu\theta + n\theta)_m, \\
 B_{\pm n} &= \frac{1}{48\pi} \sum_m \beta_{xm}^{1/2} \beta_{ym} (k_2 l)_m \cos(\psi_{\pm} - \nu_{\pm}\theta + n\theta)_m \quad (2)
 \end{aligned}$$

represent the main structural resonances. Sextupoles are considered as point-like objects with the normalized integrated strength  $(k_2 l)_m$ , and the values subscripted by " $\pm$ " have the form  $\psi_{\pm} = \psi_x \pm 2\psi_y$ , etc. Due to the TME-cell reflection symmetry (Fig.1), the above Hamiltonian expansion contains cosine terms only.

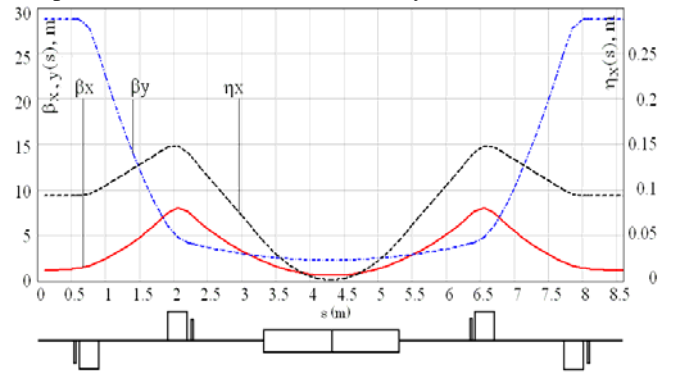


Figure 1: Schematic layout of the TME cell.

The strength of the sextupole magnets compensating natural chromaticity ( $\xi_x, \xi_y$ ) is determined by the equations

$$4\pi\xi_x + \sum_m (k_2 l)_m \beta_{xm} \eta_m = 0, \quad 4\pi\xi_y + \sum_m (k_2 l)_m \beta_{ym} \eta_m = 0, \quad (3)$$

where  $\eta(s)$  is the horizontal dispersion function. Below we consider the  $N$ -th resonant harmonics for which the cosine phase in (2) varies very slowly. Starting with  $A_{1N}$  and take into account  $(\nu_x - N)\theta \approx 0$ , we can obtain the following relation (the subscript  $N$  is omitted)

$$A_1 \approx \frac{1}{48\pi} \sum_m [\beta_x^{3/2} k_2 l]_m \cos\psi_x. \quad (4)$$

For further simplification we consider the well-known  $H$ -function

$$H = \gamma_x \eta^2 + 2\alpha_x \eta \eta' + \beta_x \eta'^2.$$

As the dispersion function obeys the free betatron oscillations equation outside the bending magnets, in this region  $H = \text{const}$  just like the Courant-Snyder invariant and we can use a Floquet transformation for the dispersion function [7]

$$H_e = a^2 + b^2,$$

$$a = \sqrt{H_e} \cos\psi_x, \quad b = \sqrt{H_e} \sin\psi_x,$$

where  $\psi_x$  is the betatron phase advance. Substituting  $\sqrt{\beta_x}/\eta = 1/\sqrt{H_e} \cos\psi_x$  in (4) and taking into account

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$H(s) = H_e = \text{const}$  (where  $H_e$  is the function value at the edge of the bending magnet) we obtain

$$A_1 \cong \frac{1}{48\pi\sqrt{H_e}} \sum_m (k_2 l)_m \beta_{xm} \eta_m, \quad (5)$$

and with (3) the following simple estimation for the resonant harmonic  $A_{1m}$  can be found

$$A_1 \cong -\frac{1}{12} \frac{\xi_x}{\sqrt{H_e}}. \quad (6)$$

A similar relation can be obtained for the harmonic  $B_{1N}$  by substituting  $-\xi_y$  instead of  $\xi_x$  (from the second equation in (3)). Assuming the betatron phase advance small (which is correct for the compact TME-cell):  $\Delta\psi_{x,y} \ll 1$ , we can obtain the following estimation for all fundamental harmonics

$$A \cong -\frac{1}{12} \frac{\xi_x}{\sqrt{H_e}}, \quad B \cong \frac{1}{12} \frac{\xi_y}{\sqrt{H_e}}. \quad (7)$$

It is necessary to remind, however, that the dynamic aperture size depends not only on the magnitude of harmonics but also on the betatron tune point. In general, such dependence is not known but near strong resonances the dynamic aperture can be found via consideration of the corresponding separatrix and fixed points. For example, in the simplest case of the resonance  $3\nu_x = N$  excited by the harmonic  $A_{3N}$ , which restricts the horizontal motion only, position of an unstable on-axis fixed point ( $x_u \neq 0, p_{xu} = 0$ ) is given by

$$x_u = -\frac{\nu_x - N/3}{3A_{3N}} \sqrt{\beta_{x0}} \cong \frac{4(\nu_x - N/3)\sqrt{H_e}}{\xi_x} \sqrt{\beta_{x0}}, \quad (8)$$

where  $\beta_{x0}$  are the betatron functions at the observation point. Similar relations can be found for other structural sextupole resonances and though these expressions might be much more complicated than (8), especially for 2D resonances, the DA size for all cases is also proportional to  $\sqrt{H_e}/\xi_{x,y}$ .

## RESONANT HARMONICS AND HORIZONTAL EMITTANCE

The equilibrium horizontal emittance in an electron storage ring is given by the balance between quantum diffusion and radiation damping, which yields

$$\varepsilon_x = \frac{C_q \gamma^2}{J_x \rho} \langle H(s) \rangle_m, \quad (9)$$

where  $C_q = 3.84 \times 10^{-13} \text{ m}$ ,  $J_x$  is the horizontal partition number,  $\rho$  is the magnetic field curvature radius, and  $\langle H(s) \rangle_m$  is the  $H$ -function averaged over the bending magnets. Taking  $\beta(s), \alpha(s), \eta(s)$  and  $\eta'(s)$  for a focusing-free bending magnet, with the length  $L$  and the bending angle  $\theta$  we obtain

$$\langle H \rangle = \frac{1}{\beta_0} \left[ \left( \eta_0 - \frac{L\theta}{24} \right)^2 + \frac{L^2 \theta^2}{720} + \frac{\beta_0^2 \theta^2}{12} \right], \quad (10)$$

$$H_e = \frac{1}{\beta_0} \left[ \left( \eta_0 - \frac{L\theta}{8} \right)^2 + \frac{\beta_0^2 \theta^2}{4} \right]. \quad (11)$$

where  $\beta_0$  and  $\eta_0$  are the values of the betatron and dispersive functions at the bending magnet center. By minimizing (10) a well-know expression for the TME minimum emittance  $\varepsilon_{x0 \min}$  can be found for particular values of  $\beta_{0 \min}$  and  $\eta_{0 \min}$  at the bending magnet center.

Using the following scaling

$$\beta_r = \frac{\beta_0}{\beta_{0 \min}}, \quad \eta_r = \frac{\eta_0}{\eta_{0 \min}} \quad \text{and} \quad \varepsilon_r = \frac{\varepsilon_{x0}}{\varepsilon_{x0 \min}} \quad (12)$$

one can find the ratio between  $H_e$  and  $\langle H \rangle$

$$H_e = g \cdot \langle H \rangle, \quad g = \frac{5(\eta_r - 3)^2 + 12\beta_r^2}{5(\eta_r - 1)^2 + 4\beta_r^2 + 4}. \quad (13)$$

Now, considering the resonance harmonics (7) and taking as a hypothesis the general expression for the boundary of the stable motion (8), we obtain

$$A_x \sim \frac{\sqrt{H_e}}{\xi_x} \sqrt{\beta_{x0}} \sim \frac{\sqrt{g \cdot \langle H \rangle}}{\xi_x} \sqrt{\beta_{x0}} \sim \frac{\sqrt{g}}{\xi_x} \cdot \sqrt{\varepsilon_x \beta_{x0}} \sim \frac{\sqrt{g}}{\xi_x} \cdot \sigma_{x0} \quad (14)$$

where  $\sigma_{x0}$  is the beam size at the DA observation point. The factor of proportionality in (14) depends on the betatron tune point ( $\nu_x, \nu_y$ ) and damping characteristics.

The function  $g(\varepsilon_r)$  may be written with the help of [8], where the relations

$$\beta_r = \varepsilon_r, \quad \eta_r = 1 + \frac{2}{\sqrt{5}} \sqrt{\varepsilon_r^2 - 1},$$

are derived subject to the condition  $\eta_r(\beta_r) = \max$ . Then

$$g = \frac{2y^2 - y\sqrt{5} + 4}{y^2 + 1}, \quad y = \sqrt{\varepsilon_r^2 - 1}.$$

It can be seen from the plot of  $\sqrt{g(\varepsilon_r)}$  that in the practical range of emittance detuning, starting from  $\varepsilon_r \sim 1.5$ , this function does not deviate significantly from its average value, by  $\sim \pm 10\%$  only, which means that the main contribution to the DA shrink with the horizontal emittance decrease (if the above hypothesis is true) is provided by beam size reduction and natural chromaticity increase.

Moreover, when measured in the units of the standard beam size  $\sigma_{x0}$ , the DA basically depends on the natural chromaticity (beside, of course, the explicit dependence of the proportionality factor on the betatron tunes, which yields a zero dynamic aperture at strong sextupole resonances).

## COMPUTER SIMULATION

In order to prove the results of analytical estimation, a computer simulation of the TME cell has been performed. Quadrupole magnets varied betatron phase advance in the range of  $0 \div 2\pi$  and two families of sextupole magnets

corrected natural chromaticity at each tune point. Particle tracking for 500 turns has been performed and the DA size was obtained. During the betatron tune scan, the following values were calculated and stored: natural chromaticity and strength of sextupole magnets necessary to correct it; horizontal emittance; five basic resonance driving terms found from the Fourier analysis of the tracking results; DA size and shape.

Fig.2 shows the natural chromaticity behavior (for one lattice cell) as a function of the horizontal betatron tune for a fixed value of vertical tune. Hereinafter the grey band around the value  $\nu_x = 0.5$  indicates a region optically unstable due to the half-integer resonance. An increase in horizontal focusing results in growth of chromaticity.

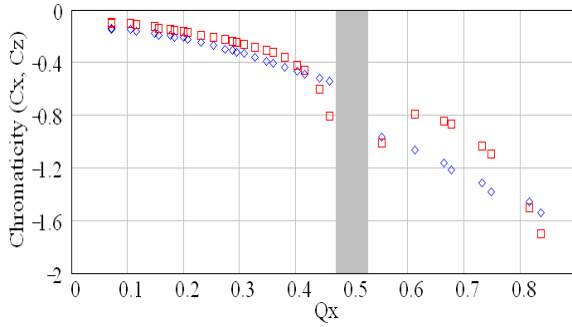


Figure 2: Natural horizontal (box) and vertical (diamond) chromaticity of the TME cell.

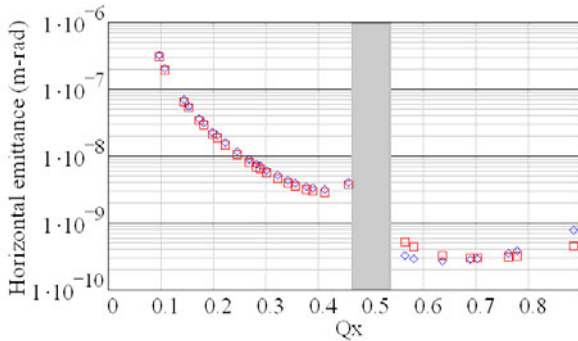


Figure 3: Horizontal emittance: simulation and estimation with (9) – (10).

The horizontal emittance as a function of the horizontal betatron phase advance is shown in Fig.3. The focusing strength increase enables reduction of emittance by  $\sim 3$  orders of magnitude, and the minimum is attained in the vicinity of  $\nu_x \approx 0.66$ . With further focusing strengthening, the emittance keeps growing, demonstrating emittance minimization relation (10).

Now, to validate our numerical simulation against the analytical model (14), we construct the following function

$$A_{xc} = \text{const} \cdot \frac{\sigma_{x0}}{\xi_x}$$

and plot it against the simulated horizontal DA. One can see in Fig.4 reasonable correspondence of two plots except for the area of structural resonances. It means that the (horizontal) DA size in the TME lattice is really

reduced with the beam size and with the natural chromaticity growing up. Strong sextupole resonances modify this behavior by its geometry factors (separatrix), which reduce the dynamic aperture additionally.

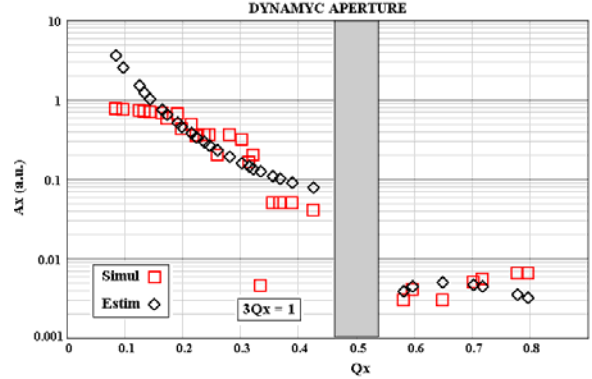


Figure 4: Horizontal DA from computer simulation (diamonds) and that estimated with (14) (boxes).

## CONCLUSIONS

Nonlinear features of the TME-cell lattice have been considered with accentuation on the relation between the strength of nonlinear perturbation due to chromatic sextupoles and fundamental characteristics of lattices (betatron phase advance, horizontal emittance and natural chromaticity). The estimation has shown that basically the dynamic aperture is proportional to the rms beam size and inversely proportional to the natural chromaticity. This general pattern of dynamic aperture is influenced by the fine structure of resonance lines, which provides additional reduction of the area of particle stable motion.

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