NEW CLASSES OF UNIFORM DISTRIBUTIONS FOR CHARGED PARTICLES IN LONGITUDINAL MAGNETIC FIELD

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Abstract

The problem of stationary self-consistent distributions for charged particles beam in longitudinal magnetic field was considered in various works. The simplest known distribution is Brillouin flow. Another simple case is Kapchinskij-Vladimirskij distribution [1]. The supporters of these distributions in the phase space of transverse configurations and velocities have zero volume. The distributions with non-zero phase volume for beams with constant cross-section radius were also obtained previously [2] – [6].

In the present report more general case is investigated when radius of beam cross-section, longitudinal velocity, and magnetic field change along the longitudinal axis. Wide classes of new stationary axially symmetric self-consistent distributions are found. New distributions have uniform charge density in the beam cross-section and, generally speaking, nonzero phase volume in the phase space of transverse motion.

In particular case of longitudinal uniformity they coincide with known ones. Such distributions can be applied for modelling of the beam in nonuniform along its axis magnetic field with particles moving with different velocities in various cross-sections. [1]

1 DYNAMICS EQUATIONS

Consider axially-symmetric stationary charged particles beam in longitudinal magnetic field. We will look for such particle distributions that particle density in configuration space $\rho(r,z)$ is constant throughout the beam cross-section:

$$\rho(r,z) = \begin{cases} \rho_0(z), r \le R(z), \\ 0, \qquad r > R(z) \end{cases}$$

where r, φ, z are cylindrical coordinates, axis z coincides with the beam axis.

Suppose that R essentially changes only at the distances which are sufficiently greater then R. Then the equation of radial motion of particles will be

$$\ddot{r} = -\omega^2 r + M^2/r^3 \qquad (1)$$

where $\omega^2 = \omega_0^2 - \lambda/R^2$, $\omega_0^2 = eB_z/2m_0\gamma$, $\lambda = eJ/2\pi\varepsilon_0 m_0\gamma^3\dot{z}$, J is beam current, ε_0 is electric constant, $\dot{z}$ is longitudinal velocity of particles supposed to be equal for all particles in given cross-section, but depended on z, $M = r^2(\dot{\varphi} + \omega_0)$, e and m_0 are charge and rest mass of particle, B_z is z-component of applied magnetic field, γ is reduced particle energy, $\gamma = (1-\beta^2)^{-1/2},\ \beta = \dot{z}/c$, dot means differentiating on independent variable t, $t \ge t_0$.

It can be shown that the equation for beam envelope $R(z)$ can be written in the form

$$\ddot{R} = -\omega^2 R + \frac{a_0^2 c_0^2}{R^3}. \qquad (2)$$

The system of equation (1) and (2) can be reduced to known Ermakov system [7] if the variable ω, which depends on t and R, is regarded as function of t.

Using the known expression for the integral of the Ermakov system [8] we obtain that the value

$$I = (R\dot{r} - r\dot{R})^2 + \frac{M^2R^2}{r^2} + \frac{a_0^2c_0^2r^2}{R^2} =$$

$$(\frac{dq}{d\tau})^2 + \frac{M^2}{q^2} + a_0^2c_0^2q^2 \qquad (3)$$

is integral of motion. Here $q = r/R$, $d\tau = dt/R^2$. Another integral of motion is

$$M = q^2(\frac{d\varphi}{d\tau} + R^2\omega_0). \qquad (4)$$

Let find such set Ω in the space of variables I, M that the condition $q \le 1, \forall t \ge t_0$ is satisfied for all particles. It follows from (3) that

$$I \le M^2 + a_0^2c_0^2. \qquad (5)$$

Also $I \ge \min\limits_q(\frac{M^2}{q^2} + a_0^2c_0^2q^2) = 2|M|a_0c_0$. Excluding the particles on the low boundary of the set we obtained that

$$I > 2|M|a_0c_0. \qquad (6)$$

The set Ω defined by the conditions (5),(6) is shown on fig.1. Analogous set for beam, uniform along its longitudinal axis, was considered in works [5], [6].

Let consider also the set Ω_q of such I and M that particle possessing these I and M passes through a point with coordinate q. First, we note that

$$I \ge \frac{M^2}{q^2} + a_0^2c_0^2q^2. \qquad (7)$$

Besides that, we have the inequality (5) limiting the value of I at a given value of M. So, the set Ω_q is defined by (7), (5).

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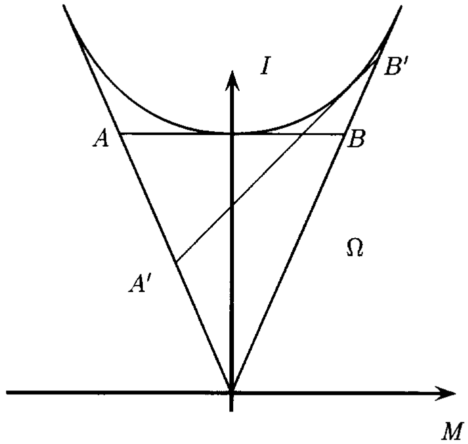

Figure 1: the set Ω.

2 DISTRIBUTION IN THE SPACE OF INTEGRALS

Further we consider the phase distribution of particles of some infinitely thin layer moving along the axis z with the velocity $\dot{z}$. We suppose that this layer is limited by two parallel infinitely closed planes moving along the axis z with the same velocity. Taking into account velocity variation along the axis and corresponding variation of the thickness of the layer we normalize all densities dividing them by the $dz/\dot{z}$.

Let us denote the particles distribution density on the variables $a, b, \ldots$ by $DN/D(a, b, \ldots)$. Consider the phase density in the four-dimensional phase space of transverse configurations and velocities $n = DN/D(x, y, \dot{x}, \dot{y})$. Here x, y are transverse Cartesian coordinates.

We assume that phase density n depends only on I and M: $n = n(I(r, \dot{r}, \dot{\varphi}), M(r, \dot{\varphi}))$ where $n(I, M)$ denotes some function of I and M. Independence from the variable φ means axially symmetry of the beam. Independence from the variable r narrows the class of admissible distribution, but sufficiently simplifies the further analysis because in that case the conservation of phase density along particles trajectories means conservation $n(I, M)$ along z-axis.

Then we have

$$\frac{DN}{D(q, I, M)} = \int_0^{2\pi} d\varphi \frac{DN}{D(q, \varphi, I, M)} = \frac{2\pi}{q^2 |q'|} \times$$

$$\frac{DN}{D(q, \varphi, q', \varphi')} = \frac{2\pi}{q^2 |q'| R^4} \frac{DN}{D(q, \varphi, \dot{q}, \dot{\varphi})} = \frac{2\pi n(I, M)}{|q'|}$$

(stroke denotes differentiating on variable τ).

Let introduce the distribution density of the particles in the space of integrals I and $M : f(I, M)$. Taking into account previous equality we get

$$f(I, M) = \int_{q_{min}(I,M)}^{q_{max}(I,M)} \frac{DN}{D(q, I, M)} \, dq = \frac{\pi^2 n(I, M)}{a_0 c_0},$$

as

$$\int_{q_{min}(I,M)}^{q_{max}(I,M)} dq/|q'| = \pi/2 a_0 c_0.$$

Expressing the particles density in configuration space $\rho(r)$ through $f(I, M)$ we get

$$\rho(r) = \int_{\Omega_q} \frac{DN}{D(r, I, M)} \frac{dI \, dM}{2\pi r} = \frac{1}{rR} \int_{\Omega_q} \frac{n(I, M) \, dI \, dM}{|q'|}.$$

Concerning the particle density in the space with the coordinates x/R, y/R denoted by $\tilde{\rho}$ we will obtain

$$\tilde{\rho}(q) = \frac{a_0 c_0}{\pi^2 q} \int_{\Omega_q} \frac{f(I, M) \, dI \, dM}{|q'|}. \tag{8}$$

In the expression (8) we exclude the particles for which $q' \equiv 0$ in accordance with (6). Accounting of these particles requires an additional term in the expression (8).

The expressions (5),(6),(8) are analogous to the expressions obtained in the work [6], where the beam with constant radius R was considered. So, results of that work can be extended to the present case.

For example, if $f(I, M)$ is the density of a simple layer[2] on the segment A'B' which is tangent to the upper boundary of the set Ω, then

$$f(I, M) = f_0 \delta_{I = I_0(k) + kM}, \; f_0 > 0, \; (I, M) \in \Omega \tag{9}$$

where $I_0(k) = a_0^2 c_0^2 - k^2/4$. Substituting (9) to (8) we have

$$\tilde{\rho}(q) = \frac{a_0 c_0 f_0}{\pi^2 q} \int_{\Omega_q} \frac{\delta_{I = I_0(k) + kM} \, dI \, dM}{(I - M^2/q^2 - a_0^2 c_0^2 q^2)^{1/2}} =$$

$$\frac{a_0 c_0 f_0 \sqrt{1 + k^2}}{\pi} \quad \text{as}$$

$$\int \delta_s \, dI \, dM = \int (\int \frac{\delta(I - I(M))}{|\cos \varphi|} dI) dM$$

where $|\cos \varphi| = (1 + k^2)^{-1/2}$. Hence, particles density throughout the beam cross-section is constant for distribution (9) and, therefore, it is solution of the problem.

Normalizing as pointed out above we have $\tilde{\rho}_0 = J/\pi$, so $f_0 = J/a_0 c_0 (1 + k^2)^{1/2}$. Here J is beam current supposed to be not depended on t.

As it is shown in the work [6], the supporter of the distribution density (9) is segment of straight line which is tangent to the upper boundary of the set Ω. The segment is

[2]The density of the simple layer is surface density of the distribution which supporter is some surface.

bounded by the lines $I = \pm 2a_0c_0M$ (the segment A'B' on the figure). When $k = 0$, this segment is parallel to the axis M (segment AB on the figure). If R is constant

$$R^2 = (\lambda + \sqrt{\lambda^2 + 4\omega_0^2 a_0^2 c_0^2})/2\omega_0^2, \quad \lambda, \omega_0 = const,$$

then the distribution (9) at $k = 0$ coincides with the Kapchinskij-Vladimirskij distribution for the beam with constant R (see [1]), and the distributions with $k \neq 0$ coincide with distributions described in [2] for beam with constant R (equilibrium of rigid rotator type).

3 LINEAR COMBINATIONS OF PARTICLES DISTRIBUTIONS

Besides that, every linear combination of the distributions (9) also will be uniform in the beam cross section:

$$f(I,M) = \sum_{k\in K} \alpha_k \delta_{I=I_0(k)+kM}$$

where $K \subset (-2a_0c_0, 2a_0c_0)$ is some set of real numbers or

$$f(I,M) = \int\limits_{-2a_0c_0}^{2a_0c_0} \alpha(k)\delta_{I=I_0(k)+kM}\, dk.$$

For these cases we have

$$\rho = \frac{a_0c_0}{\pi R^2} \sum_{k\in K} \alpha_k\,(1+k^2)^{1/2} \qquad \text{and}$$

$$\rho = \frac{a_0c_0}{\pi R^2} \int\limits_{-2a_0c_0}^{2a_0c_0} \alpha(k)\,(1+k^2)^{1/2}\, dk$$

correspondingly.

4 INTEGRAL EQUATION FOR DISTRIBUTION DENSITY

Another way to search the uniform distributions is to consider the equality (8) as integral equation for distribution density. Transposing the equation (8) we get

$$\frac{a_0^2c_0^2}{2\pi} \int\limits_0^{2\pi} \int\limits_0^1 \frac{F(y\cos(\psi-\vartheta), y\cos(\psi+\vartheta))}{(1-y^2)^{1/2}} y\,dy\,d\psi = J.$$

Here
$F(k_1,k_2) = \begin{cases} f(I,M)(M^2 - I + a_0^2c_0^2)^{1/2}, k_1 \geq k_2, \\ F(k_2,k_1), k_1 < k_2 \end{cases}$,
$k_{1,2} = 2(M \pm (M^2 - I + a_0^2c_0^2)^{1/2})$, $\vartheta = \arccos q$. This is the integral equation for the function of two arguments $F(k_1,k_2)$. Both arguments depend on q. The problem is to find such $F(k_1,k_2)$ that result of integration doesn't depend on q. Any nonnegative symmetric solution corresponds to some self-consistent particle distribution.

The simplest solution $F(k_1,k_2) = g_0$ corresponds to

$$f(I,M) = g_0(M^2 - I + a_0^2c_0^2)^{-1/2}, \quad g_0 > 0. \qquad (10)$$

Here g_0 is constant. This distribution doesn't reduce to any distributions obtained before. Another simple case is $F(k_1,k_2) = g(k_1) + g(k_2)$ corresponding to

$$f(I,M) = \frac{g(k_1)+g(k_2)}{(M^2 - I + a_0^2c_0^2)^{1/2}}, \quad g(k) \geq 0.$$

Other solutions can be sought in the form of a series or a polynomials. For example, the distribution

$$f(I,M) = \frac{-c\,(I - a_0^2c_0^2)(10M^2 - 5I + 2a_0^2c_0^2) + g_0}{(M^2 - I + a_0^2c_0^2)^{1/2}}$$

corresponds to the polynomial of third degree. The constant values c and g_0 must be taken so that $f(I,M) \geq 0$ for all $(I,M) \in \Omega$.

Thus, wide classes of new stationary self-consistent nonuniform along z-axis distributions are obtained. These distributions can be used for the solution of various problems of calculation and optimization of accelerating structures.

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