REVIEW OF IMAGE EFFECTS FOR PARTICLE BEAMS IN CYLINDRICAL PIPES

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Abstract

The effects of a perfectly conducting beam pipe are examined for both a centered continuous beam and a centered bunched beam. A set of coupled, ordinary differential equations is derived for each case. These equations describe the dynamics of the rms beam envelopes along the design trajectory. Finally, an example application is presented for the bunched beam situation.

1 INTRODUCTION

For some time we have been investigating the image effects from a cylindrical beam pipe on beam dynamics, in particular for space-charge dominated beams [1,2,3]. By image effects we mean, of course, the action of the charges induced on the pipe from the beam's space charge. Here we review the most useful conclusions of these investigations. Specifically, sets of ordinary differential equations are presented which describe the dynamics of the rms beam envelopes. These results are essentially the continuation of work initiated by Sacherer in the early 1970's [4].

1.1 Ellipsoidal Symmetry and Equivalent Beams

We restrict our attention to particle beams having elliptical symmetry for continuous beams and ellipsoidal symmetry for bunched beams. The results are applicable to quadrupole focusing systems for the continuous beam case (and, of course, solenoidal systems). However, we further restrict our attention to the axisymmetric situation in the bunched beam case. Since we are primarily interested in the longitudinal effects, this simplifies the analysis so that most of the results can be expressed analytically in terms of elementary functions (otherwise, elliptical integrals would appear in the final system). The longitudinal results still apply in an average sense to bunched beams having quadrupole symmetry.

All the results presented here are expressed in terms of an equivalent uniform beam. As Sacherer discovered, the statistical dynamics of the beam (i.e., the rms envelopes) are only loosely coupled, if at all, to the actual distribution of the beam. Therefore, we choose our model to be the uniform beam, since it has well-defined envelopes (unlike a gaussian or a thermal distribution). The equivalent beam principle tells us that the actual beam may be modeled by a uniform beam so long as both beams have the same second spatial moments and rms emittances. The differential equations given here describe the beam envelopes of the equivalent uniform beam.

1.2 Further Limitions

The major short-coming of the analysis is that is does not describe a self-consistent situation. Specifically, our model for the self-fields of the beam does not consistently couple with the beam dynamics. The net result is that there is no model for rms emittance growth in the beam. However, in the space-charge dominated situation this drawback is of little concern. In this case the beam dynamics are dominated by the self-fields of the beam and thermal effects play only a minor role. Moreover, these beams tend toward a uniform distribution and it has beam shown that in the bunched beam case that the equilibrium (stationary) distribution is closely approximated by an ellipsoid [5].

1.3 Moment Equations

The general idea of the analysis is to take moments of the equations of motion with respect to the particle beam distribution. The Vlasov equation then allows the time derivative operator and the moment operator to commute. The net result is a set of coupled, ordinary differential equations involving the moments. We concentrate on the second order moments. Letting $\langle\cdot\rangle$ denote the moment operator with respect to the particle beam distribution, we find the following [6]:

$$\langle x^2\rangle'' - \frac{[\langle x^2\rangle']^2}{2\langle x^2\rangle} + \frac{2k_x(\zeta)}{\gamma m v^2}\langle x^2\rangle - \frac{2q}{\gamma^3 m v^2}\langle xE_x\rangle - \frac{2\tilde{\epsilon}_x^{\,2}}{\langle x^2\rangle} = 0\,,$$
$$\langle y^2\rangle'' - \frac{[\langle y^2\rangle']^2}{2\langle y^2\rangle} + \frac{2k_y(\zeta)}{\gamma m v^2}\langle y^2\rangle - \frac{2q}{\gamma^3 m v^2}\langle yE_y\rangle - \frac{2\tilde{\epsilon}_y^{\,2}}{\langle y^2\rangle} = 0\,,$$
$$\langle z^2\rangle'' - \frac{[\langle z^2\rangle']^2}{2\langle z^2\rangle} + \frac{2k_z(\zeta)}{\gamma m v^2}\langle z^2\rangle - \frac{2q}{\gamma m v^2}\langle zE_z\rangle - \frac{2\tilde{\epsilon}_z^{\,2}}{\langle z^2\rangle} = 0\,.$$

where ζ is the path length along the design trajectory, the prime indicates differentiation with respect to ζ, γ is the relativistic factor, v is the bunch velocity (assumed constant), q and m are the particles' charge and mass, respectively, the $k_\alpha(\zeta)$'s are the "spring constants" of the focusing system, the $\tilde{\epsilon}_\alpha$'s are the rms emittances, and the E_α are the self electric fields. In the above system we have assumed that the individual magnetic self-fields in the beam frame are negligible but the collective magnetic field is not. Our task is now consigned to the determination of the moments $\langle \alpha E_\alpha\rangle$.

2 CONTINUOUS BEAMS

Here we assume that the beam is continuous along the

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design trajectory and that the ζ variations are slow enough to allow for an accurate 2D transverse plane analysis. To find the moment $\langle xE_x\rangle$ with images we employed a Green's function technique. By expanding Green's function in a trigonometric series (in polar coordinates), we can identify the self-field terms corresponding to the induced (image) charges on the beam pipe. Taking the moments of these fields we get [2]

$$\langle xE_x\rangle = \frac{Q}{4\pi\epsilon_0}\left[\frac{\langle x^2\rangle^{1/2}}{\langle x^2\rangle^{1/2}+\langle y^2\rangle^{1/2}} + \frac{2\langle x^2\rangle}{b^4}\left(\langle x^2\rangle - \langle y^2\rangle\right)\right] + O\left(\frac{\langle x^2\rangle^2}{b^8}\right),$$

where b is the radius of the beam pipe, Q is the charge per cross-section, and $O(\cdot)$ indicates the standard order notation. The first term was originally calculated by Sacherer and is the free-space self-field contribution. The second term represents the contribution due to the induced charges from the distribution's quadrupole moment. The last term is meant to say that the image forces due to the higher order moments of the charge distribution (octupole moments and up) scale as $\langle x^4\rangle^2/b^8$, etc. Note that the above expression is independent of the distribution, this condition leads to the notion of equivalent beams.

Substituting the above equation and an analogous one for the y direction into the moment equations includes the lowest order effects from images. Translating the resulting moment equations into the equations for the equivalent uniform beam yields the result

$$X'' + \kappa_x(\zeta)X - \frac{2K}{X+Y} - \frac{\epsilon_x^2}{X^3} - \frac{K}{4b^4}(X^3 - XY^2) = 0,$$
$$Y'' + \kappa_y(\zeta)Y - \frac{2K}{X+Y} - \frac{\epsilon_y^2}{Y^3} - \frac{K}{4b^4}(Y^3 - YX^2) = 0,$$

where $X(\zeta)$ and $Y(\zeta)$ are the beam envelopes, the $\kappa_\alpha(z)=k_\alpha(z)/(\gamma m v^2)$ are the focusing functions, and $K=qI/(2\pi\epsilon_0\gamma m v^3)$ is the generalized beam perveance [7] (I being the beam current). The quantities ϵ_x and ϵ_y are the effective emittances of the beam given by $4\tilde{\epsilon}_x$ and $4\tilde{\epsilon}_y$, respectively. These equations are recognized as the standard KV coupled-envelope equations with the addition of a term (for each equation) accounting for the dominant image effects. Since the beam is centered and has elliptical symmetry, $\langle x^3\rangle$ is zero and the next image term will be an octupole term.

One may see that the image effects do not play a large role in the beam dynamics. However, for beams with large eccentricities we have seen this role to be significant. Matching sections, overall, will probably be more susceptible to image effects since the beam envelopes tend to make larger excursions through them.

3 BUNCHED BEAMS

As mentioned in the introduction we restrict our attention to the axisymmetric case and use cylindrical coordinates (r,z) where $r^2=x^2+y^2$. Therefore, the bulk of the analysis is the determination of the moments $\langle rE_r\rangle$ and $\langle zE_z\rangle$. Although for the free-space situation these moments may be determined analytically in terms of elementary functions, they are not simple expressions. Therefore, we simply present the results of the analysis for the equivalent uniform ellipsoid. The interested reader is referred to [3] for the details. We have

$$R'' + \kappa_r(\zeta)R - \frac{3K}{4\sqrt{5}}\frac{1}{R^2}W_r\left(\frac{Z}{\sqrt{2}R}\right) - \frac{45K}{2}\frac{R\cdot A_f(R,Z)}{(Z^2-R^2)^{3/2}} - \frac{\epsilon_r^2}{R^3} = 0$$
$$Z'' + \kappa_z(\zeta)Z - \frac{3\gamma^2 K}{2}\frac{1}{Z^2}W_z\left(\frac{Z}{\sqrt{2}R}\right) + \frac{45\gamma^2 K}{2}\frac{Z\cdot A_f(R,Z)}{(Z^2-R^2)^{3/2}} - \frac{\epsilon_z^2}{Z^3} = 0$$

where $R(\zeta)$ and $Z(\zeta)$ are the beam envelopes, and ϵ_r and ϵ_z are the effective emittances given by $5/2\tilde{\epsilon}_r$ and $5\tilde{\epsilon}_z$, respectively. The functions W_r and W_z come from the free-space self-fields and the function A_f represents the image field effects on the dynamics. This function depends upon the actual distribution profile of the particle beam, indicated by the label f. The function $A_f(R,Z)$ actually has a simpler representation as given below.

$$A_f(R,Z) \equiv \begin{cases} A_f\left(\dfrac{b}{\sqrt{Z^2-R^2}}\right) & \text{for } Z > R, \\ 0 & \text{for } Z < R, \end{cases}$$

(The zero value for A_f comes from the fact that image effects are negligible whenever Z<R.) Thus, we see that A_f really depends upon only one parameter. The graphs of A_f for several different distributions are shown in Fig. 2. The expression for W_r and W_z are given below.

$$W_r(s) \equiv \begin{cases} \dfrac{1}{2}\dfrac{1}{(\frac{1}{2}-s^2)^{3/2}}\arctan\dfrac{\sqrt{\frac{1}{2}-s^2}}{s} - \dfrac{s}{\frac{1}{2}-s^2} & \text{for } s\in[0,\frac{1}{\sqrt{2}}), \\ \dfrac{s}{s^2-\frac{1}{2}} - \dfrac{1}{2}\dfrac{1}{(s^2-\frac{1}{2})^{3/2}}\operatorname{arctanh}\dfrac{\sqrt{s^2-\frac{1}{2}}}{s} & \text{for } s\in(\frac{1}{\sqrt{2}},\infty), \end{cases}$$

and

$$W_z(s) \equiv \begin{cases} \dfrac{s^2}{\frac{1}{2}-s^2} - \dfrac{s^3}{(\frac{1}{2}-s^2)^{3/2}}\arctan\dfrac{\sqrt{\frac{1}{2}-s^2}}{s} & \text{for } s\in[0,\frac{1}{\sqrt{2}}), \\ \dfrac{s^3}{(s^2-\frac{1}{2})^{3/2}}\operatorname{arctanh}\dfrac{\sqrt{s^2-\frac{1}{2}}}{s} - \dfrac{s^2}{s^2-\frac{1}{2}} & \text{for } s\in(\frac{1}{\sqrt{2}},\infty). \end{cases}$$

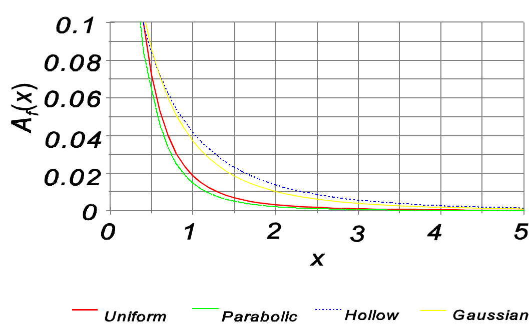


Figure 1: Function $A_f(x)$ for different distributions

In the space-charge dominated situation the uniform distribution is of primary importance. The expression for A_f in this case is given below

$$A_f(x) = \int_{-\infty}^{+\infty} \frac{K_0(|\omega|x)}{I_0(\omega x)} \left(\frac{3\cos\omega}{\omega^2} - \frac{3\operatorname{sinc}\omega}{\omega^2} + \operatorname{sinc}\omega \right) \left(\frac{\cos\omega}{\omega^2} - \frac{\operatorname{sinc}\omega}{\omega^2} \right) d\omega .$$

This expression may be integrated numerically to form a table of values for $A_f(x)$ (for interpolation). Once done the differential system may be integrated using standard numerical techniques. We may also express in terms of an infinite series of Bessel functions

$$\begin{aligned} A_f(x) &= -\frac{61}{450}\pi - \frac{\pi}{15}\ln\left(\frac{x}{4}\right) \\ &+ \sum_{n=1}^{\infty} \frac{2\pi}{\alpha_n^2 J_1^2(\alpha_n)} \left(\frac{x}{\alpha_n} + \frac{x^2}{\alpha_n^2} \right) \\ &\cdot \left(\frac{x}{\alpha_n}\sinh\frac{\alpha_n}{x} - \frac{3x^2}{\alpha_n^2}\cosh\frac{\alpha_n}{x} + \frac{3x^3}{\alpha_n^3}\sinh\frac{\alpha_n}{x} \right) e^{-\frac{\alpha_n}{x}} . \end{aligned}$$

This expression is convenient for asymptotic analysis.

3.1 A Simple Example

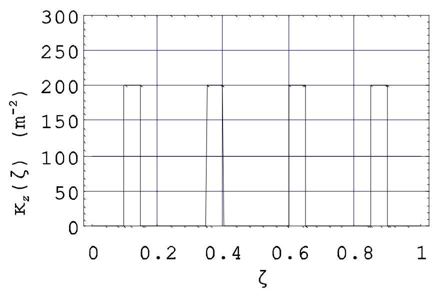
To illustrate the utility of these results we have simulated a simple transport system for bunched particle beams. The transport system has uniform focusing in the radial direction and periodic focusing in the longitudinal direction. The focusing functions for the z direction is shown in Fig. 2a. We have used a “hard-edge” function for $\kappa_z(\zeta)$. The period of $\kappa_z(\zeta)$ is 25cm, the pulse length is 5cm, with maximum value 200m^{-2}. The constant value of $\kappa_r(\zeta)$ is 100m^{-2}. The beam parameters are given as follows: K=0.01, ϵ_r=5×10^{-5}m-rad, ϵ_z=2×10^{-5}m-rad, and γ=1.

The matched beam solution in free space is shown in Fig. 2b while the matched beam solution including a pipe with radius b=5cm is shown in Fig. 2c. In both situations the top curves is the Z envelope. Obviously, the pipe has a substantial effect on the beam dynamics in this case. Note the coupling between the radial and longitudinal motion. Both planes oscillate synchronously, in phase with the longitudinal envelope equation, as they travel down the beam axis.

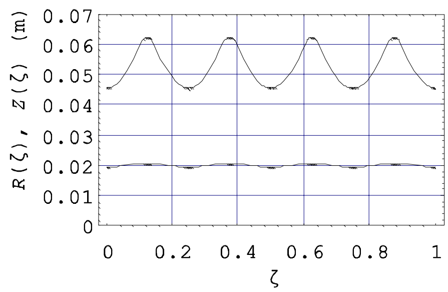
4 CONCLUSIONS

In the continuous beam case the image effects are relatively minor but can become a factor if the beam undergoes extreme eccentricities . It would seem to be sound policy to design systems which minimizes image effects and the extended KV equations offer a simple formula to analytically determine when they may become troublesome.

In the bunched beam case image effects become unavoidable whenever $Z>R$. The designer must account for these effects in beam transport systems. The differential equations presented herein provide a way to simulate bunch behavior, at least in the space-charge dominated regime.



a)



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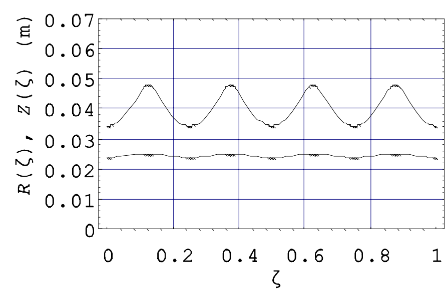

c)

Figure 2: Bunched beam example

5 REFERENCES

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