WAVELET APPROACH TO ACCELERATOR PROBLEMS, I. POLYNOMIAL DYNAMICS

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Abstract

This is the first part of a series of talks in which we present applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In the general case we have the solution as a multiresolution expansion in the base of compactly supported wavelet basis. The solution is parametrized by solutions of two reduced algebraical problems, one is nonlinear and the second is some linear problem, which is obtained from one of the next wavelet constructions: Fast Wavelet Transform, Stationary Subdivision Schemes, the method of Connection Coefficients.

In this paper we consider the problem of calculation of orbital motion in storage rings. The key point in the solution of this problem is the use of the methods of wavelet analysis, relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (including pseudodifferential) in such bases. Our problem as many related problems in the framework of our type of approximations of complicated physical nonlinearities is reduced to the problem of the solving of the systems of differential equations with polynomial nonlinearities with or without some constraints. In this paper we consider as the main example the particle motion in storage rings in standard approach. Starting from Hamiltonian, which described classical dynamics in storage rings and using Serret–Frenet parametrization, we have after standard manipulations with truncation of power series expansion of square root the corresponding equations of motion:

$$\begin{aligned}
&\frac{d}{ds}x = \frac{p_x + H\cdot z}{[1+f(p_\sigma)]}; \qquad (1)\\
&\frac{d}{ds}p_x = \frac{[p_z - H\cdot x]}{[1+f(p_\sigma)]}\cdot H - [K_x^2 + g]\cdot x + N\cdot z\\
&+K_x\cdot f(p_\sigma) - \frac{\lambda}{2}\cdot(x^2-z^2) - \frac{\mu}{6}(x^3 - 3xz^2);\\
&\frac{d}{ds}z = \frac{p_z - H\cdot x}{[1+f(p_\sigma)]};\\
&\frac{d}{ds}p_z = -\frac{[p_x + H\cdot z]}{[1+f(p_\sigma)]}\cdot H - [K_z^2 - g]\cdot z\\
&+N\cdot x + K_z\cdot f(p_\sigma) - \lambda\cdot xz - \frac{\mu}{6}(z^3 - 3x^2 z);\\
&\frac{d}{ds}\sigma = 1 - [1 + K_x\cdot x + K_z\cdot z]\cdot f'(p_\sigma) -\\
&\frac{1}{2}\cdot\frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma)\\
&\frac{d}{ds}p_\sigma = -\frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right]
\end{aligned}$$

Then we use series expansion of function $f(p_\sigma)$ and the corresponding expansion of RHS of equations (1). In the following we take into account only an arbitrary polynomial (in terms of dynamical variables) expressions and neglecting all nonpolynomial types of expressions, i.e. we consider such approximations of RHS, which are not more than polynomial functions in dynamical variables and arbitrary functions of independent variable s ("time" in our case, if we consider our system of equations as dynamical problem). The first main part of our construction is some variational approach to this problem, which reduces initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. We consider also two private cases of our general construction. In the first case (particular) we have for Riccati equations (particular quadratic approximations) the solution as a series on shifted Legendre polynomials, which is parameterized by the solution of reduced algebraical (also Riccati) system of equations. This is only an example of general construction. In the second case (general polynomial system) we have the solution in a compactly supported wavelet basis. Multiresolution expansion is the second main part of our construction. The solution is parameterized by solutions of two reduced algebraical problems, one as in the first case and the second is some linear problem, which is obtained from one of the next wavelet construction: Fast Wavelet Transform (FWT), Stationary Subdivision Schemes (SSS), the method of Connection Coefficients (CC). Our problems may be formulated as the systems of ordinary differential equations $dx_i/dt = f_i(x_j,t), \quad (i,j=1,...,n)$ with fixed initial conditions $x_i(0)$, where f_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time. Because of time dilation we can consider only next time interval: $0\le t\le 1$. Let us consider a set of functions $\Phi_i(t) = x_i dy_i/dt + f_i y_i$ and a set of functionals $F_i(x) = \int_0^1 \Phi_i(t)dt - x_i y_i\mid_0^1$, where $y_i(t)(y_i(0)=0)$ are dual variables. It is obvious that the initial system and the system $F_i(x)=0$ are equivalent. In part 3 we consider

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symplectization of this approach. Now we consider formal expansions for x_i, y_i:

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t) \quad y_j(t) = \sum_r \eta_j^r \varphi_r(t), \quad (2)$$

where because of initial conditions we need only $\varphi_k(0) = 0$. Then we have the following reduced algebraical system of equations on the set of unknown coefficients λ_i^k of expansions (2):

$$\sum_k \mu_{kr}\lambda_i^k - \gamma_i^r(\lambda_j) = 0 \quad (3)$$

Its coefficients are $\mu_{kr} = \int_0^1 \varphi_k'(t)\varphi_r(t)dt$, $\gamma_i^r = \int_0^1 f_i(x_j,t)\varphi_r(t)dt$. Now, when we solve system (3) and determine unknown coefficients from formal expansion (2) we therefore obtain the solution of our initial problem. It should be noted if we consider only truncated expansion (2) with N terms then we have from (3) the system of $N \times n$ algebraical equations and the degree of this algebraical system coincides with degree of initial differential system. So, we have the solution of the initial nonlinear (polynomial) problem in the form

$$x_i(t) = x_i(0) + \sum_{k=1}^N \lambda_i^k X_k(t), \quad (4)$$

where coefficients λ_i^k are roots of the corresponding reduced algebraical problem (3). Consequently, we have an parametrization of solution of initial problem by solution of reduced algebraical problem (3). But in general case, when the problem of computations of coefficients of reduced algebraical system (3) is not solved explicitly as in the quadratic case, which we shall consider below, we have also parametrization of solution (4) by solution of corresponding problems, which appear when we need to calculate coefficients of (3). As we shall see, these problems may be explicitly solved in wavelet approach. Next we consider the construction of explicit time solution for our problem. The obtained solutions are given in the form (4), where in our first case we have $X_k(t) = Q_k(t)$, where $Q_k(t)$ are shifted Legendre polynomials and λ_k^i are roots of reduced quadratic system of equations. In wavelet case $X_k(t)$ correspond to multiresolution expansions in the base of compactly supported wavelets and λ_k^i are the roots of corresponding general polynomial system (3) with coefficients, which are given by FWT, SSS or CC constructions. According to the variational method to give the reduction from differential to algebraical system of equations we need compute the objects γ_a^j and μ_{ji}, which are constructed from objects:

$$\begin{aligned}
\sigma_i &\equiv \int_0^1 X_i(\tau)d\tau = (-1)^{i+1}, \quad (5)\\
\nu_{ij} &\equiv \int_0^1 X_i(\tau)X_j(\tau)d\tau = \sigma_i\sigma_j + \frac{\delta_{ij}}{(2j+1)},\\
\mu_{ji} &\equiv \int X_i'(\tau)X_j(\tau)d\tau = \sigma_j F_1(i,0) + F_1(i,j),\\
& F_1(r,s) = [1-(-1)^{r+s}]\hat{s}(r-s-1),\\
& \hat{s}(p) = \begin{cases} 1, & p \geq 0 \\ 0, & p < 0 \end{cases}\\
\beta_{klj} &\equiv \int_0^1 X_k(\tau)X_l(\tau)X_j(\tau)d\tau = \sigma_k\sigma_l\sigma_j + \\
& \alpha_{klj} + \frac{\sigma_k\delta_{jl}}{2j+1} + \frac{\sigma_l\delta_{kj}}{2k+1} + \frac{\sigma_j\delta_{kl}}{2l+1},\\
\alpha_{klj} &\equiv \int_0^1 X_k^* X_l^* X_j^* d\tau = \\
& \frac{1}{(j+k+l+1)R\big(1/2(i+j+k)\big)} \times \\
& R\big(1/2(j+k-l)\big)R\big(1/2(j-k+l)\big) \times \\
& R\big(1/2(-j+k+l)\big),
\end{aligned}$$

if $j+k+l = 2m, m \in Z$,and $\alpha_{klj} = 0$ if $j+k+l = 2m+1$; where $R(i) = (2i)!/(2^i i!)^2$, $Q_i = \sigma_i + P_i^*$, where the second equality in the formulae for $\sigma, \nu, \mu, \beta, \alpha$ hold for the first case. Now we give construction for computations of objects(5) in the wavelet case. We use some constructions from multiresolution analysis: a sequence of successive approximation closed subspaces V_j: $...V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset ...$ satisfying the following properties: $\bigcap_{j\in\mathbf{Z}} V_j = 0$, $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$, $f(x) \in V_j <=> f(2x) \in V_{j+1}$ There is a function $\varphi \in V_0$ such that $\{\varphi_{0,k}(x) = \varphi(x-k)_{k\in\mathbf{Z}}\}$ forms a Riesz basis for V_0. We use compactly supported wavelet basis: orthonormal basis for functions in $L^2(\mathbf{R})$. As usually $\varphi(x)$ is a scaling function, $\psi(x)$ is a wavelet function, where $\varphi_i(x) = \varphi(x-i)$. Scaling relation that defines φ, ψ are

$$\begin{aligned}
\varphi(x) &= \sum_{k=0}^{N-1} a_k\varphi(2x-k) = \sum_{k=0}^{N-1} a_k\varphi_k(2x),\\
\psi(x) &= \sum_{k=-1}^{N-2} (-1)^k a_{k+1}\varphi(2x+k)
\end{aligned}$$

Let be $f: \mathbf{R} \longrightarrow \mathbf{C}$ and the wavelet expansion is

$$f(x) = \sum_{\ell\in\mathbf{Z}} c_\ell\varphi_\ell(x) + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} c_{jk}\psi_{jk}(x) \quad (6)$$

The indices k, ℓ and j represent translation and scaling, respectively

$$\varphi_{jl}(x) = 2^{j/2}\varphi(2^j x - \ell), \psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$$

The set $\{\varphi_{j,k}\}_{k\in\mathbf{Z}}$ forms a Riesz basis for V_j. Let W_j be the orthonormal complement of V_j with respect to V_{j+1}. Just as V_j is spanned by dilation and translations of the scaling function, so are W_j spanned by translations and dilation of the mother wavelet $\psi_{jk}(x)$. If in formulae (6) $c_{jk} = 0$ for $j \geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only $f(x) = \sum_{\ell\in\mathbf{Z}} c_{J\ell}\varphi_{J\ell}(x)$. This is a finite wavelet expansion, it can be written solely in terms of translated scaling functions. We use wavelet $\psi(x)$, which has k vanishing moments $\int x^k\psi(x)d(x)=0$, or equivalently $x^k=\sum c_\ell\varphi_\ell(x)$ for each k, $0\le k\le K$. Also we have the shortest possible support: scaling function DN (where N is even integer) will have support $[0,N-1]$ and $N/2$ vanishing moments. There exists $\lambda>0$ such that DN has λN continuous derivatives; for small $N,\lambda\ge 0.55$. To solve our second associated linear problem we need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$. Let be $\varphi^n_\ell = d^n\varphi_\ell(x)/dx^n$. We derive the wavelet - Galerkin approximation of a differentiated $f(x)$ as $f^d(x)=\sum_\ell c_l\varphi^d_\ell(x)$ and values $\varphi^d_\ell(x)$ can be expanded in terms of $\varphi(x)$

$$\phi^d_\ell(x)=\sum_m\lambda_m\varphi_m(x),\quad \lambda_m=\int\limits_{-\infty}^{\infty}\varphi^d_\ell(x)\varphi_m(x)dx$$

The coefficients λ_m are 2-term connection coefficients. In general we need to find

$$\Lambda^{d_1d_2\dots d_n}_{\ell_1\ell_2\dots\ell_n}=\int\limits_{-\infty}^{\infty}\prod\varphi^{d_i}_{\ell_i}(x)dx \tag{7}$$

For Riccati case we need to evaluate two and three connection coefficients

$$\begin{aligned}\Lambda^{d_1d_2}_\ell &= \int_{-\infty}^{\infty}\varphi^{d_1}(x)\varphi^{d_2}_\ell(x)dx,\quad d_i\ge 0,\\ \Lambda^{d_1d_2d_3} &= \int\limits_{-\infty}^{\infty}\varphi^{d_1}(x)\varphi^{d_2}_\ell(x)\varphi^{d_3}_m(x)dx\end{aligned}$$

According to CC method [7] we use the next construction. When N in scaling equation is a finite even positive integer the function $\varphi(x)$ has compact support contained in $[0,N-1]$. For a fixed triple (d_1,d_2,d_3) only some $\Lambda^{d_1d_2d_3}_{\ell m}$ are nonzero : $2-N\le\ell\le N-2,\quad 2-N\le m\le N-2,\quad |\ell-m|\le N-2$. There are $M=3N^2-9N+7$ such pairs (ℓ,m). Let $\Lambda^{d_1d_2d_3}$ be an M-vector, whose components are numbers $\Lambda^{d_1d_2d_3}_{\ell m}$. Then we have the first key result: Λ satisfy the system of equations

$$\begin{aligned}A\Lambda^{d_1d_2d_3} &= 2^{1-d}\Lambda^{d_1d_2d_3},\ d=d_1+d_2+d_3,\\ A_{\ell,m;q,r} &= \sum_p a_pa_{q-2\ell+p}a_{r-2m+p}\end{aligned}$$

By moment equations we have created a system of $M+d+1$ equations in M unknowns. It has rank M and we can obtain unique solution by combination of LU decomposition and QR algorithm. The second key result gives us the 2-term connection coefficients:

$$\begin{aligned}A\Lambda^{d_1d_2} &= 2^{1-d}\Lambda^{d_1d_2},\quad d=d_1+d_2,\\ A_{\ell,q} &= \sum_p a_pa_{q-2\ell+p}\end{aligned}$$

For nonquadratic case we have analogously additional linear problems for objects (7). Also, we use FWT and SSS for computing coefficients of reduced algebraic systems. We use for modelling D6,D8,D10 functions and programs RADAU and DOPRI for testing.

As a result we obtained the explicit time solution (4) of our problem. In comparison with wavelet expansion on the real line which we use now and in calculation of Galerkin approximation, Melnikov function approach, etc also we need to use periodized wavelet expansion, i.e. wavelet expansion on finite interval. Also in the solution of perturbed system we have some problem with variable coefficients. For solving last problem we need to consider one more refinement equation for scaling function $\phi_2(x)$: $\phi_2(x)=\sum_{k=0}^{N-1}a^2_k\phi_2(2x-k)$ and corresponding wavelet expansion for variable coefficients $b(t)$: $\sum_k B^j_k(b)\phi_2(2^jx-k)$, where $B^j_k(b)$ are functionals supported in a small neighborhood of $2^{-j}k$.

The solution of the first problem consists in periodizing. In this case we use expansion into periodized wavelets defined by $\phi^{per}_{-j,k}(x)=2^{j/2}\sum_Z\phi(2^jx+2^j\ell-k)$. All these modifications lead only to transformations of coefficients of reduced algebraic system, but general scheme remains the same. Extendeed version and related results may be found in [1]-[6].

This research was supported in part by the “New Ideas for Particle Accelerator Programs” under the National Science Foundation Grant No. PHY94-07194.

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