# "INTEGRABLE ACCELERATOR" STRUCTURE FOR ROUND BEAMS WITH A SOLITON-LIKE FORCE

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*Abstract*

This work is devoted to finding the integrable systems for accelerators with strong nonlinear fields. Among all the solutions there is one with a soliton-like force; one of the invariants for this system is a quartic in momentum polynomial. Another one is angular momentum, so this system is related with round beams. Such a field can be made by solenoidal focusing field, so the soluthion presents an example of integrable accelerator with regular nonlinear motion.

All this shows the relation of general theory of integrable systems with particular theory of accelerators; it might initiate applications of Lie groups to equations in partial derivatives for finding "integrable" lattices of accelerators.

## 1 INTRODUCTION

In high energy circular aceelerators and storage rings the betatron motion of particles can be perfectly described by 2D Hamiltonian, whose form corresponds to the usual non-relativistic motion with a time-dependent force. For such a case there is no possibility to analyse motion of particles by analytical formulas because of the stochasticity of trajectories in phase space. Only the special Hamiltonians give regular motion. For this we need, for example, 2 commuting integrals of motion.

In this paper it is shown, that system with presented below Hamiltonian has 2 commuting integral of motion. Hamiltonian $H$ is (for simplicity particle's mass is equal to 1):

$$H = p_x^2/2 + p_y^2/2 + f_1(t)\cdot r^2/4 + g(r)/4. \qquad (1)$$

where $r = \sqrt{x^2+y^2}$ and $f_1$ satisfies

$$f_1''' + 8\cdot f_1 \cdot f_1' - 3\cdot h\cdot f_1 = 0.$$

This reminds the equation for a traveling wave solution $f(x-ct)$ of the Korteweg–de Vries equation:

$$f''' + f\cdot f' - c\cdot f' = 0.$$

After changing $f_1$ to $f/8$ and $3h$ to $c$ in the equation for $f_1$, we come to the last one.

Hamiltonian (1) depends only on the radius $r = \sqrt{x^2+y^2}$, so one of the integrals is the angular momentum. In the next section we'll show, how to construct 1D systems with polynomial in momentum integrals of motion, and show, that the presented above system has an additional quartic in momentum invariant.

## 2 1D DYNAMICAL SYSTEMS WITH INVARIANTS, POLYNOMIAL IN MOMENTUM

Further we use the method of finding integrable systems, proposed by Whittaker [1].

Following [2], we construct here a family of continuous time-dependent 1D Hamiltonians which have a quadratic invariant, and thus the respective motion in 1.5D is integrable. Consider a general form of invariant quadratic in momentum $p$, assuming that the coefficients $A,\ B,\ V$ are arbitrary functions of time $t$ and coordinate $x$:

$$I = \frac{1}{2}(Ap-B)^2 + V\,,$$

$A \neq 0$. Equating its total time derivative to zero, we account for the Hamiltonian equations of motion $\dot{x} = p$, $\dot{p} = f$, where the unknown force $f$ depends on $t,\ x$

$$\frac{dI}{dt} = (Ap-B)\Big(A_x p^2 + (A_t - B_x)p + Af - B_t\Big)$$
$$+V_x p + V_t \equiv 0\,,$$

the subscripts here denote the respective partial derivatives. The vanishing coefficients of each power of $p$ yield a set of equations in partial derivatives for the four unknown functions: $f$ is to be found along with $A,\ B,\ V$.

First of all, $A_x = 0$, and $A = A(t)$ is an arbitrary function of time. Then we take $A_t - B_x = 0$, whence:

$$B(x,t) = \dot{A}x + A^2\dot{D}\,,$$

with an arbitrary $D(t)$; dots denote the time derivatives. We choose here the special form of arbitrary additive function of time for future convenience.

The last two equations form a set of equations specifying the unknowns $f$ and $V$:

$$A(Af - B_t) + V_x = 0 \qquad (2)$$
$$-B(Af - B_t) + V_t = 0 \qquad (3)$$

The force $f$ is thus expressed via $V,\ B,\ A$:

$$f = -\frac{V_x}{A^2} + \frac{B_t}{A}\,, \qquad (4)$$

and $V$ is determined by the homogeneous equation in partial derivatives of the 1st order:

$$B\,V_x + A\,V_t = 0$$

Its characteristic curve $x(t)$ is then obtained from the equation:

$$\frac{dx}{dt} = \frac{B}{A} = \frac{\dot{A}x + A^2\dot{D}}{A}\,.$$

Integration gives the lines of constant level of $V$:

$$X \equiv \frac{x}{A(t)} - D(t) = const\,.$$

Hence,

$$V(x,t) = U\Big(\frac{x}{A(t)} - D(t)\Big)\,,$$

where $U(X)$ is an arbitrary function.

Thus we conclude, that the general solution to our problem of integrable system in 1.5D with quadratic invariant is generated with three arbitrary functions: $A(t)$, $D(t)$, and $U(X)$. The corresponding force $f$ is then found from (4):

$$f = -\frac{1}{A^3}U' + \frac{1}{A}(\ddot{A}x + (A^2\dot{D})^{\cdot})\,.$$

The Hamiltonian of this system can be found from the expression for $f$:

$$\mathcal{H}(x,p,t) = \frac{p^2}{2} + \frac{1}{A^2}U(\frac{x}{A} - D) - \frac{A\ddot{A}}{2}(\frac{x}{A})^2 - (A^2\dot{D})^{\cdot}\,\frac{x}{A}\,,$$

and the final form of the desired invariant is:

$$I = \frac{1}{2}(Ap - \dot{A}x - A^2\dot{D})^2 + U(\frac{x}{A} - D)\,.$$

It's possible to find by this method invariants of arbitrary order in momentum. It's shown in [3], that cubic in momentum invariants and corresponding forces may be found from linear equations also (like in quadratic in momentum case).

### *2.1 Integrable systems for round beams*

So now we know, how to construct 1D integrable systems. What is the difference between a common 1D case and 2D systems with conservation of the angular momentum? Actually a 2D system with the angular momentum conservation can be reduced to a 1D system with the 'centrifugal' force $\mathcal{M}^2/x^3$. But the difference is in the condition, that the second invariant for this system must exist for each value of $\mathcal{M}$. For $\mathcal{M} = 0$ we have a usual 1D system, so the desired integrable system for round beams gives a common 1D integrable motion, but in general the converse statement is not true (an additional invariant may exist only for zero angular momentum, for example). So the family of integrable systems for round beams is less than the family of common 1D integrable systems.

### *2.2 Integrable systems with soliton-like forces*

Let's take the simplest quartic polynomial as an invariant:

$$I = p^4 + A(x,t)\cdot p^2 + B(x,t)\cdot p + C(x,t);$$

here we omit the cubic terms at all, $A, B, C$ are unknown functions. After differentiation of this invariant w.r.t. to time we should have zero coefficients of each power of $p$ (since $dI/dt = 0$), so we have the following set of equations:

$$\begin{aligned} 4F + \frac{\partial A}{\partial x} &= 0 \\ \frac{\partial A}{\partial t} + \frac{\partial B}{\partial x} &= 0 \\ 2\cdot A\cdot F + \frac{\partial B}{\partial t} + \frac{\partial C}{\partial x} &= 0 \\ BF + \frac{\partial C}{\partial t} &= 0, \end{aligned} \tag{5}$$

where $F = \dot{p}$ is the force.

Let's suppose, that we have found some solution of these equations. Now we want, that $F + \mathcal{M}^2/x^3$ be also a solution of the previous set of equations. From the first line of (5) we see, that $A$ has to be transformed into $A + 2\mathcal{M}^2/x^2$, and $B$ stays unchanged.[1]

Let's eliminate the function $C$ from the equations. We can just take the partial derivative of the third equation in (5) w.r.t. to $t$, and the partial derivative of the forth one w.r.t. to $x$, and subtract one from another. We have:

$$2\frac{\partial(A\cdot F)}{\partial t} + \frac{\partial^2 B}{\partial t^2} = \frac{\partial(B\cdot F)}{\partial x}.$$

Let's put here the new $A$ and $F$. We have:

$$2(A_t\cdot F + A\cdot F_t) + 4\mathcal{M}^2/x^2\cdot F_t + 2\mathcal{M}^2/x^3\cdot A_t + B_{tt} =$$
$$B_x(F + \mathcal{M}^2/x^3) + B(F_x - 3\mathcal{M}^2/x^4);$$

here $A, B, F$ are the old functions independent of $\mathcal{M}$, subscripts $t, x$ mean partial derivatives w.r.t. to $t, x$.

We want to get solutions for an arbitrary value of the angular momentum, so we demand, that each coefficient at any power of $\mathcal{M}$ should vanish. So, from the previous equation we obtain two ones (without $\mathcal{M}$); adding to them the first two equations of (5) we have:

$$\begin{aligned} 4F + A_x &= 0 \\ A_t + B_x &= 0 \\ 3A_t\cdot F + 2A\cdot F_t + B_{tt} &= B\cdot F_x \\ 4F_t\cdot x^2 + 2A_t\cdot x &= B_x\cdot x - 3B. \end{aligned} \tag{6}$$

This is a set of four equations for three unknown functions. A check for consistency is needed in general, in order to verify the existence of any solutions.

To eliminate now $F$ and $B$, we take $F$ from the first and $B_x$ from the second equation and substitute these in the last one (having taken its time derivative). After that we get an equation for $A$:

$$x^2 A_{xxt} - xA_{xt} = 0.$$

All the solutions are:

$$A = f_1(t)x^2 + f_2(t) + g(x);$$

[1] This is not the only way of transformation of $A$ and $B$. For example, $A$ may get a term, which depends on time and momentum. But here we are looking for the simplest solutions.

here $f_1, f_2, g$ are free functions.

After that we have to use one more equation for $A, B, F$, namely the third one in the set (6). At first, let's express $F$ and $B$ using the solution for $A$. From the first equation of (6) we have:

$$4F = -2f_1(t) \cdot x - g'(x),$$

and from the second and last ones we have:

$$3B = -3(f_1'(t) \cdot x^3 + f_2'(t) \cdot x) + 2f_1'(t) \cdot x^3.$$

Let's put $f_2 = 0$ for saving calculations (the case $f_2 \neq 0$ can be treated in the same way, but with more complications). Then $B = -f_1' x^3/3$.

The third equation of (6) now reads:

$$-3f_1'(t) \cdot x^2 \cdot \frac{2f_1(t) \cdot x + g'(x)}{4} - (f_1(t) \cdot x^2 + g(x)) f_1'(x) \cdot x$$

$$-f'''(t) \cdot x^3/3 = f_1'(t) \cdot x^3 \cdot \frac{2f_1(t) + g''(x)}{12}.$$

We rewrite it in a more compact form:

$$(-8/3 \cdot f_1(t) f_1'(t) - f_1'''(t)/3) \cdot x^3 =$$

$$f_1'(t)(3g'(x)x^2/4 + g''(x) \cdot x^3/12 + g(x) \cdot x).$$

Having thus separated the variables, we now have:

1. $f_1(t) = const$, $g$ is an arbitrary function of $x$. It's the case of the invariant quadratic in momentum; now we have it squared.

2.

$$3g'(x)x^2/4 + g''(x) \cdot x^3/12 + g(x) \cdot x = -h \cdot x^3, \quad (7)$$

where $h$ is an arbitrary constant. For $f_1(t)$ we have:

$$f_1''' + 8 \cdot f_1 \cdot f_1' - 3 \cdot h \cdot f_1 = 0. \quad (8)$$

This reminds the equation for a traveling wave solution $f(x - ct)$ of the Korteweg–de Vries equation:

$$f''' + f \cdot f' - c \cdot f' = 0.$$

After changing $f_1$ to $f/8$ and $3h$ to $c$ in (8), we come to this equation. Then we can find $g$ from (7). Our force then is equal to

$$F(x, t) = -f_1(t) \cdot x/2 - g'(x)/4.$$

So, the presented above method gives an analytical tool aimed at finding integrable systems for round beams (or, even for systems with more complicated integrals of motion, than the angular momentum).

## 3 ACKNOWLEDGEMENTS

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## 4 REFERENCES

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