

line can be defined by six parameters (see Fig. 1); first, we execute a global translation of the component, then, three rotations, leaving invariant the middle point of the displaced entry face of the component (L'_m in Fig. 1):

- the horizontal, vertical and longitudinal displacements dx , dy and ds respectively.
- the rotation θ around the actual vertical axis, \hat{y}_{in} in Fig. 1, or yaw angle, the rotation ϕ around the newly defined horizontal axis, \hat{x}'_{in} in Fig. 1, or pitch angle, the rotation ψ around the newly defined longitudinal axis or roll angle.

We will forget here the “longitudinal alignment errors”, ds and ψ , which do not create any transverse offset³. Then, we consider a component of length L with the misalignments dx and θ , and we suppose that the beam goes into the component (abscissa s_0) with the initial conditions $\mathcal{X}(s_0)$; using the previous section, we can compute the beam offset at the component exit ($s_1 = s_0 + L$) in the following way:

$$\mathcal{X}(s_1) = \mathcal{R}(s_0 \rightarrow s_1) \left(\mathcal{X}(s_0) + \mathcal{X}_{in}(dx, \theta) \right) + \mathcal{X}_{out}(dx, \theta)$$

where $\mathcal{X}_{in}(dx, \theta) = -[dx, \theta, \dots, dx, \theta]$ and $\mathcal{X}_{out}(dx, \theta) = [dx + L\theta, \theta, \dots, dx + L\theta, \theta]$ refers to the change of coordinate at the component entry and at the component exit respectively. Thus, at any abscissa s_f , we can estimate the line sensitivity relative to the misalignments of any of its components; for instance, for a yaw angle θ_i of the element number i , we obtain:

$$\partial_{\theta_i} \mathcal{X}(s_f) = \mathcal{R}(s_i + L_i \rightarrow s_f) \left(\Theta_{out}(L_i) + \mathcal{R}(s_i \rightarrow s_i + L_i) \Theta_{in} \right)$$

with $\Theta_{in} = -[0, 1 \dots 0, 1]$ and $\Theta_{out}(L_i) = [L_i, 1 \dots L_i, 1]$. Finally, in the same way, we can obtain all the response coefficients of the line, concerning the initial offset (angle and position) of the bunch.

4 CORRECTION ALGORITHMS

The simplest alignment methods are based on a correction of the bunch centre-of-charge deviation measured at one or several beam position monitors (BPM) within the line by moving the quadrupoles transversely. More recently, new correction schemes have been proposed, based on the simultaneous minimisation of trajectory deviations and trajectory differences linked to variable quadrupole settings or linked to variable bunch charge or bunch length [2]. However that may be, all these correction techniques have in common the following fact: they all come down to the minimisation by moving quadrupoles of a certain function Φ which depends quadratically on measurements of trajectory deviations. Then, it can be shown [3] that after this minimisation the vector $\mathcal{X}(s_f)$ (containing the positions and angles of the n_s slices at the line output) does not depend anymore on the quad misalignments before correction but

³For flat beams, the roll angle ψ , coupling both transverse planes, may become critical for the emittance growth in the plane where the beam dimension is the smallest, but, actually, it is not yet implemented in the method.

only and in a linear way on the errors of the line⁴, that we write in the formal following way:

$$\mathcal{X}(s_f) \stackrel{\text{def}}{=} M_{er} dX_{er} \quad (6)$$

where dX_{er} is a stochastic vector containing the “errors of the line” and where M_{er} is a $2n_s \times N_{er}$ matrix which depends on the focusing lattice (via the response coefficients) and on the choice of the correction algorithm (via the choice of the function Φ).

5 STATISTICAL BEAM RESPONSE IN TERM OF EMITTANCE GROWTH

From now on the rest of the paper, the notation $\langle Q \rangle_{er}$ will represent the average of the quantity Q over the statistic distribution of the “errors”, compared to the same notation, without suffix, referring to an average over the longitudinal particle distribution $\rho(z)$. So, we suppose that the vector, $\mathbf{m}_{er} \equiv \langle dX_{er} \rangle_{er}$, and the covariance matrix, $\Gamma_{er} \equiv \langle dX_{er}^T dX_{er} \rangle_{er} - \mathbf{m}_{er}^T \mathbf{m}_{er}$, are known quantities which depends on the pre-alignment scheme⁵.

For the sake of simplification, we will assume $\mathbf{m}_{er} = 0$. By using Eq. 6, we can compute the average and the covariance matrix of the vector $\mathcal{X}(s_f)$ at the line output:

$$\begin{aligned} \tilde{\mathbf{m}} &\stackrel{\text{def}}{=} \langle \mathcal{X}(s_f) \rangle_{er} = M_{er} \mathbf{m}_{er} = 0 \\ \tilde{\Gamma} &\stackrel{\text{def}}{=} \langle \mathcal{X}(s_f)^T \mathcal{X}(s_f) \rangle_{er} = M_{er} \Gamma_{er}^T M_{er} \end{aligned} \quad (7)$$

Nevertheless, for reasons which will appear later, we prefer to define the symmetric matrix $\Gamma_{1 \leq i \leq 2n_s, 1 \leq j \leq 2n_s}$ by:

$$\begin{cases} \Gamma_{2i-1, 2j-1} = \Delta Q_{ij} \left\langle (\delta x(z_i) - \langle \delta x \rangle) (\delta x(z_j) - \langle \delta x \rangle) \right\rangle_{er} \\ \Gamma_{2i, 2j} = \Delta Q_{ij} \left\langle (\delta x'(z_i) - \langle \delta x' \rangle) (\delta x'(z_j) - \langle \delta x' \rangle) \right\rangle_{er} \\ \Gamma_{2i-1, 2j} = \Delta Q_{ij} \left\langle (\delta x(z_i) - \langle \delta x \rangle) (\delta x'(z_j) - \langle \delta x' \rangle) \right\rangle_{er} \\ 1 \leq i \leq n_s, 1 \leq j \leq n_s, \text{ where } \Delta Q_{ij} \stackrel{\text{def}}{=} \Delta z \sqrt{\rho(z_i) \rho(z_j)}, \\ \langle \delta x \rangle = \sum_{n=1}^{n_s} \Delta z \rho(z_n) \delta x(z_n) \text{ and } \langle \delta x' \rangle = \sum_{n=1}^{n_s} \Delta z \rho(z_n) \delta x'(z_n) \end{cases}$$

which can be easily computed from $\tilde{\Gamma}$ and $\rho(z)$. The beam matrix of the whole bunch at the line output is then given by: $\mathcal{S} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix}$ where the notations used are explained hereafter.

- $\langle x^2 \rangle$ is the RMS value of the transverse beam extension at the line output. Since the motions of $\tilde{x}(z)$ and $\delta x(z)$ are uncorrelated (Section 1), we have:

$$\langle x^2 \rangle = \int dz \rho(z) [(\sigma_x^2(z) + (\delta x(z) - \langle \delta x \rangle)^2)]$$

⁴From now on the rest of the paper, by “errors of the line”, we will always refer to the structure and BPM misalignments, to the BPM and correction resolution, and to the initial conditions of the different trajectories considered in the correction algorithm.

⁵Here, we will only consider static misalignments of the structures, resulting from the linac pre-alignment strategy; then, we suppose that the characteristic time required to apply the correction remains very small compared to the characteristic period of the machine vibrations (ground motion).

where $\sigma_x(z)$ represents the transverse size (RMS value) of the z -slice at the line output.

- In a similar way, and with evident notations, we have:

$$\begin{aligned} \langle x'^2 \rangle &= \int dz \rho(z) [\sigma_{x'}^2(z) + (\delta x'(z) - \langle \delta x' \rangle)^2] \\ \langle x x' \rangle &= \int dz \rho(z) [\sigma_{x x'}(z) + (\delta x(z) - \langle \delta x \rangle) (\delta x'(z) - \langle \delta x' \rangle)] \end{aligned}$$

Thus, we can write the matrix \mathcal{S} in the following manner:

$$\mathcal{S} = \Sigma_0 + \begin{pmatrix} \sum_{n=1}^{n_s} X_{2n-1}^2 & \sum_{n=1}^{n_s} X_{2n-1} X_{2n} \\ \sum_{n=1}^{n_s} X_{2n-1} X_{2n} & \sum_{n=1}^{n_s} X_{2n}^2 \end{pmatrix} \quad (8)$$

where Σ_0 is the beam matrix at the line output, assuming no error within the line⁶, and where X is the $2n_s$ -dimensional vector defined by:

$$\begin{cases} X_{2n-1} = \sqrt{\Delta z \rho(z_n)} (\delta x(z_n) - \langle \delta x \rangle), & 1 \leq n \leq n_s \\ X_{2n} = \sqrt{\Delta z \rho(z_n)} (\delta x'(z_n) - \langle \delta x' \rangle), & 1 \leq n \leq n_s \end{cases} \quad (9)$$

the covariance matrix of which is the matrix Γ previously defined. Finally, the RMS emittance is defined as usual by:

$$\epsilon(X) \stackrel{\text{def}}{=} \sqrt{\det(\mathcal{S})} \quad (10)$$

In order to go further in the computations, we are forced to make the following assumption: we assume the vector X_{er} to be Gaussian, so that the vector X becomes also Gaussian⁷. Hence, the knowledge of Γ permits the complete description of the statistical distribution ρ_X related to the vector X :

$$\rho_X(X) = \frac{1}{(2\pi)^{n_s} \sqrt{\det(\Gamma)}} \exp\left(-\frac{1}{2} X^T \Gamma^{-1} X\right) \quad (11)$$

The first and second moment of the emittance distribution are then:

$$\begin{aligned} \langle \epsilon \rangle_{er} &= \int_{\mathbf{R}^{2n_s}} \epsilon(X) \rho_X(X) dX \\ \langle \epsilon^2 \rangle_{er} &= \int_{\mathbf{R}^{2n_s}} \epsilon^2(X) \rho_X(X) dX \end{aligned} \quad (12)$$

The second moment is easy to compute, since the function $\epsilon^2(X)$ is a polynomial in X of degree 4. For the computation of the first moment, a direct estimation by numerical integration is of course out of question (the computation time growing exponentially with $2n_s$). In fact, we are able to reduce this integral to a 2-dimensional integral (independently of n_s) and, under these conditions, we can compute

⁶ $\Sigma_0 = \sum_{n=1}^{n_s} \rho(z_n) \Delta z R_{n,n}^0 \Sigma_{in}(z_n) {}^T R_{n,n}^0$ where $\Sigma_{in}(z_n)$ is the beam matrix of the slice z_n at the line input and $R_{n,n}^0$, the diagonal block (n, n) of the generalised R-matrix of the whole line, which is also the classical 2×2 transfer matrix describing the motion of the slice z_n without any wakefield (Section 1).

⁷ X depends linearly on the vector \mathcal{X} , Eq. 9, which depends linearly on the vector X_{er} , Eq. 6.

it numerically [3].

If now, we note $N(\epsilon)$, the percentage of machines which give, a final single-bunch emittance lower than ϵ , we have:

$$N(\epsilon) = \int_{\epsilon(X) \leq \epsilon} \rho_X(X) dX = \int_{\mathbf{R}^{2n_s}} \Theta(\epsilon - \epsilon(X)) \rho_X(X) dX$$

where Θ represents the Heaviside step function. Thus, by definition of the density ρ_ϵ , we have:

$$\rho_\epsilon(\epsilon) \stackrel{\text{def}}{=} \frac{dN}{d\epsilon}(\epsilon) = \int_{\mathbf{R}^{2n_s}} \delta(\epsilon - \epsilon(X)) \rho_X(X) dX \quad (13)$$

where δ is the Dirac distribution. Unfortunately, no solution was found permitting the estimation of this distribution in the most general case. Indeed, we have to approximate $\epsilon(X)$ by its development at the second order in the coefficients of the vector X . In other words, we assume that a perturbation regime has been reached, for which the non-constant part of the function $\epsilon(X)$ becomes small, when averaging over the distribution of the beam line errors, in such a way that the moments of order greater than two can be neglected. Thus, we write:

$$\epsilon(X) = \sqrt{\det(\Sigma_0)} \left[1 + \frac{1}{2} X^T \Gamma_0^{-1} X \right] + o(X^4) \quad (14)$$

where Γ_0 is the following 2×2 block diagonal matrix:

$$\Gamma_0 \stackrel{\text{def}}{=} \begin{pmatrix} \Sigma_0 & 0 & \dots & 0 \\ 0 & \Sigma_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Sigma_0 \end{pmatrix} \quad (15)$$

Finally, with this approximation, we obtain [3]:

$$\rho_\epsilon(\Delta\epsilon/\epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp(ik(\Delta\epsilon/\epsilon)) \left(\prod_{p=1}^{2n_s} (1 + ik\lambda_p) \right)^{-\frac{1}{2}}$$

where $\Delta\epsilon/\epsilon \stackrel{\text{def}}{=} (\epsilon - \sqrt{\det \Sigma_0}) / \sqrt{\det \Sigma_0}$ and where λ_p , $1 \leq p \leq 2n_s$, are the $2n_s$ eigenvalues of the matrix $\Gamma \Gamma_0^{-1}$.

6 CONCLUSION

Starting from a given probability law for the misalignments of the beam line components, a given trajectory correction scheme, and a given focusing lattice, the method computes the statistical distribution laws of certain quantities relative to the beam itself. Therefore, the power of this new approach lies in the fact that it can generate, in one single passage, statistical results which could be reproduced by a classical tracking program, but in a much more tedious way (several runs on several beam lines with randomly generated misalignments).

7 REFERENCES

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