ANALYTIC NONLINEAR METHODS FOR BEAM OPTICS

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Abstract

In recent years there have been major advances in the computation and use of high-order maps for the design, optimization and operation of beamlines. We will describe five practical examples for both linear and circular colliders.

1 INTRODUCTION

The range of meaning for the word "analytical" extends from a hand-derived formula to be used by hand, to a computer-derived formula to be used by hand (eg. with a symbolic manipulation program), to a computer-derived expression to be seen only by a computer. There is an increasing degree of complexity as one proceeds through these types. The hand-derived/hand-used is especially important for the first stages of beamline design, the computer-derived/hand-used is especially important for use in the operation and tuning of an existing beamline, and computer-derived/computer-used is useful in the intermediate and final stages of beamline design. We will describe five examples, covering a full range of complexity. From least to most complex these examples will be:

- use of similarity transformations (FFTB design, SLC diagnosis)
- statistical maps (SSC smear and tune-shift)
- aberrations (SLC upgrade)
- resonance basis and nPB tracking (PEP-II design)
- kick factorization (possibly LHC).

2 LIE OPERATOR BASICS

The Lie operator and associated algebra are valuable tools to understanding the examples to be described. We introduce them briefly here. See reference [1].

2.1 Lie operators

For R a constant, the equation $\frac{d}{dt} f = R f$ has the solution $f(t+\Delta t) = e^{\Delta t R} f(t)$. This equation could equally well be written $f(t+\Delta t) = e^{\Delta t \frac{d}{dt}} f(t)$, which is now true for a very large class of functions f(t). We will be concerned with a class of functions $f(x(s), p_x(s), ...)$ where $x(s)$ is the particle position, and the differential operator, d/ds, is given by a Hamiltonian. Symbolically $\frac{d}{ds} = -:H(x, p_x, ..):$ where $:H: f = \{H, f\}$ and where {.. ,..} is the symbol for the Posson bracket. In other words

$$f(s+\Delta s) = e^{\Delta s \frac{d}{ds}} f(s) = e^{-\Delta s :H:} f(x, p_x, ..)$$

We often drop the ": ... :" notation when no confusion can arise.

2.2 Composition laws

The usefulness of the Lie operator symbol lies in the three following properites:

1. Concatenation. If the Hamiltonian H changes abruptly from a function H_1 in a segment Δs_1 to a function H_2 in a segment Δs_2, then the result of transporting through the two adjacent segments is $e^{-\Delta s_1 :H_1:} e^{-\Delta s_2 :H_2:}$.

2. Composition (BCH law). For generator functions A and B, there is a generator C, such that $e^A e^B = e^C$ where C is given by a perturbation series of Poisson brackets: $C = A + B + \frac{1}{2}\{A, B\} + ...$

3. Similarity. For generator functions A and B, $e^A e^B e^{-A} = e^{e^A(B)}$. This law says that the simlarity transform of a Lie operator is given by the Lie operator with an approriately transformed generator. This law looks rather special, but has wide applicability.

2.3 Representation of elements and beamlines

From the concatenation law it is clear that a beamline can be represented by a product of Lie operators. This result is enhanced by the fact that element misplacements can be represented by coordinate transformations specified by Lie operators inserted between elements. Likewise fringe fields, edge angles, and overlapping fields (eg. a quadrupole sitting in a solenoid) all can be faithfully represented by the appropriate Lie operators.[2]

2.4 Dragt-Finn map representation

We have stated that any element can be represented by a Lie operator and a beamline can be represented by a product of such operators. There are a variety of means by which these operator products can be composed to find a Lie operator representation for the entire beamline. The final result usually takes the form of a product of a linear operator, specified either by a matrix or a Lie operator, and a purely nonlinear operator. More often the beamline map is determined by tracking through the elements with

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a power series to determine a power series representation of the beamline map. Using the factorizaiton result derived by Dragt and Finn any power series which represents a symplectic map can be represented by the product of a linear map and a nonlinear Lie operator. [3]

2.5 Normal forms

Normal form factorizations apply to closed rings. In rings one often seeks a representation of the form

$$M\, e^{G} = e^{A}\, \mathrm{R} e^{h}\, e^{-A}$$

where M is the linear map for the ring, G in the generator of the remainder nonlinear map, R is a block-diagonal rotation matrix, h is a generator that depends only on action operators, and A is the generator of a similarity transformation that maps the phase-space invariant surfaces of the original map, when they exist, onto a product of toroids. [4]

This normal form may be found formally but will only be accurate under limited conditions that are difficult to specify precisely. It is important to distinguish between properties of the one-turn map, which will generally be well behaved and continuous in all the variables defining it, and the "iterated map" which may have a range of behaviors, including of course, chaotic motion. Hence the normal form, which can only represent very regular behavior, can not represent the full range of expected properties of the iterated map.

3. STATISTICAL MAPS

During a design process, one must assign statistical values to many element properties, such as multipole strengths and magnet positions. One must then study an ensemble of machines to be sure that all have acceptable behavior. It would be advantageous if one could avoid the map composition process for all rings in the statistical sample, and assign statistical values directly to map coefficients. The following work describes the first foray into such territory [5]. This work also provides valuable insight into how the nonlinear generator of the one-turn map is related to the nonlinearities of beamline elements.

3.1 The motive

The SSC design group had chosen a quantity called smear, which was roughly the rms spread of the invariant action, along with tune-shift-with-amplitude, to characterize the behavior of lattices prior to studying their long-term behavior. Since these quantities depended on the random seed determining each lattice, a large computational effort was required to determine smear for a statistical sample of each basic design. Forest showed that these quantities could be found using Lie methods. Bengtsson and Irwin included closed-orbit effects and firmly established that the results of the calculation and tracking were identical, including the Fourier decomposition of the smear. Weeks of computational time were reduced to minutes.

3.2 The method

Starting with the representation of the one-turn map as a product of Lie operators, each representing an element, one first solves for corrector strengths to find a satisfactory closed orbit. For the phase advances of the SSC lattice and the location of correctors and BPMs, one could do this quite simply using a local bump algorithm. With the corrector strengths determined, operators could be introduced to represent them.

The principal step is to write each element map as a product of two linear maps bracketing a remainder containing the chromaticity and nonlinear multipole terms. Similarity transformations can now be used to move all linear maps to the front of the line. The result is that the variables in the remainder map generators are replaced by a linear sum expressing the coordinate at the center of the element location as a function of the position and momenta at the end of the beamline.

Since the remainders are small maps, the BCH law can now be used to find the nonlinear generator of the one-turn map. The first term in the BCH law is just the sum of the element remainder generators.

The apparent complexity of a 100 km ring conaining a variety of nonlinear terms spread out along the circumference is replaced by a linear matrix and one nonliner map specified by a polynomial starting with third order terms. The polynomial is given, to first order, as a sum of the polynomials of the nonlinearities in the ring, each written as a function of the position and momenta at the end of the ring. The situation could hardly be simpler!

Furthermore using the BCH theorem there is a clear prescription how to find next order terms so that they can be calculated and compared to the first order terms. This approach is a usual perturbation theory in the strength of element nonlinearity strength. Its advantage is simplicity and clarity.

Most of the multipole strengths contain a stochastic variable. The sum of stochastic variables can be represented as a stochastic variable. In this way one finds directly an expression for the map with stochastic variables in coefficients.

3.3 Smear and tune-shift-with-amplitude

One can next look at the normal form expression to find the generator of the simlarity transformation in terms of the one-turn nonlinear map generator. At this step the tune of the ring enters explicitly. Finally the generator of the tune-shift-with-amplitude can be found. There is a complication that important terms in this expression are second order in the sextupole terms of the one-turn generator.

4. KICK FACTORIZATION

A kick map, which for example might represent an impulse approximation for an element, has a generator that depends on only one phase space variable. It leaves the variable x unchanged and increments the momenta by a function of the position. A generalization of the kick would be a map whose generator was a coordinate transformation of a kick generator. These have also been referred to as a jolts.

4.1 The motive

Kick maps and products of kick maps are very easy and fast to evaulate. They are guaranteed sympletic because no truncation is required. Since the one-turn map is the result of adding together a bunch of kicks in the case where each element is represented by a symplectic kick factorization, it is natural to ask whether given a one-turn generator one could find a factorization, hopefully with a much smaller number of factors, that would faithfully represent it. If so, the one turn-map could be tracked very rapidly to study the long-term behavior so important in proton machines.

4.2 The method

One posits a set of kicks with simple, but specified phase-space rotations between them [6]. The number of kicks is determined by the number of terms required to represent the highest order polynomial.

One first solves a linear equation for the coefficients of the third order polynomials. Next one determines the fourth-order effects of these polynomials, subtracts these terms from the original one-turn map and fits the remainder by a sum of fourth-order polynomials, and so on.

4.3 Improvements

Rotations between kicks can be specified by a set of points in a 2D plane, where the x coordinate is the horizontal-plane rotation angle, and the y coordinate is the vertical-plane rotation angle. One quickly realizes that, if the points are on a coordinate grid, an unusually large number of points is required because of a degeneracy. So one is led to tilt this grid slightly. Techniques developed by Abell and Dragt [7] have shown that there is an optimal angle for this tilt. They have also studied other groups of linear maps between kicks, in search of a "best choice".

5. USE OF SIMILARITY TRANSFORMATIONS

Final focus systems for linear colliders are unusal in that the nonlinearities are necessarily very strong. If uncorrected, the chromaticity of the final doublet in the Next Linear Collider (NLC) design would give an rms size to the beam that is about 100 times larger than its linear size. This huge chromatic term is compensated by a pair of sextupoles separated by a -I section upstream from the doublet, and these chromatic kicks coming via the presence of dispersion in these sextupoles must in turn be several times smaller than the sextupole kicks themselves to insure good system bandwidth. These nonlinearities cannot be dealt with effectively, or with any insight, using the BCH theorem.

However because the sextupole nonlinearities must cancel out, there must be a structure in the beamline that insures this. This structure is, of course, the -I module between them. Placing the sextupole at each end of the -I, and removing linear terms, as described in section 3, we see that we are looking at a simlarity transformation, where the generator of the similarity transform is the sextupole generator.

5.2 Chromatic-correction section map

The remainder generator for the -I between sextupoles contain chromatic terms of the form

$$a_n x^2 \delta^n + b_n p_x^2 \delta^n + ..$$

along with similar terms in y. There are no xp_x terms if the -I section is forward-backward symmetric. The generator of the sextupole is of the form:

$$G_S = \frac{k_S}{3!}[(x \pm \eta\delta)^3 - 3(x \pm \eta\delta)y^2]$$

$$= \frac{k_S}{3!}[x^3 - 3xy^2 + 3\eta^2 x\delta^2 \pm 3\eta(x^2 - y^2)\delta + .]$$

To form the similarity transform, we add and subtract two times the chromatic term on the left hand side. This leaves a residuual chromatic term twice the size of the term in the sextupole generator, and a sturcture which is a pure similarity transform [8]. The map for this module may now be determined precisely by replacing p_x by $p_x - \frac{\partial G_S}{\partial x}$, and similarly for p_y. This transformed generator has the property that no large terms remain. The chromaticty term has been removed. It must be moved through the final telescope to cancel the chromaticity of the final doublet, producing another similarity structure to evaluate. And the sextupole terms are gone. What is left is a transform of the -I chromatic terms by the presence of the sextupoles. Since the derivative of G_S is second order, and this is squared, the highest order aberration which arises from the linear chromaticity of the -I will be of fifth order.

Besides determining an important high order aberration, which by design must be kept acceptably small, we have established that there are no additional terms. We have found a very consise expression for these aberrations which can be used in determining optimal design parameters (length, b functions, dispersion) for final focus design [9].

5.3 Similarity transformation generality

The considerations of the preceding paragraph are surprisingly general. Often one asks questions such as, "How bad can the launch conditions be?" or "How large can a misalignement be?". Since it is presumed that there will be a corrector to compensate steering, or a change in a quadrupole to compensate a mismatch, all of these questions resolve themselves in a similarity transfrom structure. [10]

6. ABERRATIONS

Each term in the nonlinear beamline generator corresponds to a unique aberration. These terms will be small by design, even for beamlines with large nonlinearities.

6.1 Final focus system upgrade

In evaluating existing, and typically non-ideal beamlines, one will use symbolic manipulator programs or truncated power series agebras to determine the beamline map. Investigation of the generator will reveal what aberrations are dominating the beamline. Modifications to eliminate the aberrations can be considered. Before a recent upgrade, the dominant aberration on the SLC final focus beamline was the term $\delta^2 p_y^2$, a second-order (in δ) chromaticity term. This term could be corrected by putting a quadrupole in the final telescope [11]. Since there is a large first order chromaticity at each end of this telescope, small phase advance changes between these terms produces the second order term. This can be seen by evaluating the Poisson bracket $\{p_{x1}^2, p_{x2}^2\} = p_{x1} p_{x2}\{p_{x1}, p_{x2}\} \approx p_{x1}^2\{p_{x1}, p_{x2}\}$.

7. RESONANCE BASIS

For rings it is more informative to write the nonlinear one-turn map generator in a resonance basis. The transform to this basis is linear, hence easy to perform. It plays an important role in the normal form theory. The generator of a ring will have many, many terms and in general it is impossible to determine the effect of an aberration by looking at its strength. However in the course of the PEPII design a graphic display was developed [12] to monitor the strengths of these aberrations. This proved useful for two reasons:
inadvertent mistakes in the input lattice, where quickly spotted, and several general lessons were learned. One important lesson concerned the strength of chromatic terms (anything quadratic in phase-space variables with a d dependence). This is of course well-known, but the required conditions became more quantitative.

8. NPB TRACKING

In the design process, many what-if questions naturally arise. What if we were able to change the strength of a particular resonance? What if we changed the tune? These questions can be answered if one can reliably track with the map.

8.1 The nPB tracking method

Because one contemplated asking questions about changes in resonance strengths, it was natural to seek a method to track in an action-angle basis. It was a great surprise to find that one could, and with excellent speed. The method consists of applying the map to the standard phase-space variables x, p_x, .. but evaulating the resulting Poisson brackets in the action-angle variables. Astonishingly this can be done without ever evaluating a sine or cosine, or square root. See reference [12] for details. Results were compared to element-by-element tracking to establish that the dynamic aperture determinations were identical.

8.2 Swimps and swamps

A SWIMP is an acronym we have given to a “switched map”. A switched map is derived from the one-turn map of a beamline by altering it in some interesting way. The most important SWIMPs used in the PEPII design process were those for which the one-turn phase advance of the linear map was changed while the nonlinear map remained unchanged. Since the phase trombone of PEPII is located in a benign section of beamline, this can correspond quite closely to a real situation. Varying the tune over one quadrant of the tune plane, and looking at the dynamic aperture, resulted in what has been called a SWAMP plot. In this way the behavior of our lattices could be studied in the entire tune plane, rather than just one point. [13]

9. SUMMARY

An example of an analytical result in nonlinear beam optics was presented for five distint levels of complexity. Starting from the lowest, these examples were:

- use of similarity transformations to determine design equations for linear collider final focus systems
- use of the BCH law to determine a statistical sample of one-turn maps for proton rings
- use of the nonlinear generator to find and correct aberrations in a beamline map, or monitor resonance strengths in a ring map
- description of an action-angle based tracking algorithm that allows one to determine dynamic apertures of electron rings in the entire tune plane
- description of a kick factorization method for a one-turn map that may be used to rapidly determine the long-term behavior of proton rings

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REFERENCES

[1] For a recent overview of this subject and bibliography we recommend A. J. Dragt, 'Maps past, present, and future'. Part. Accel. **54**, (1996)

[2] E. Forest, Mm. Reusch, D. Bruhwiler, and A. Amiry, 'The correct local description for tracking in rings', Part. Accel. **45**, 65 (1994)

[3] A. Dragt and J. Finn, 'Lie series and invariant functions for anlytic symplectic maps', J. Math. Phys. **17**, 2215 (1976)

[4] E. Forest, M. Berz, and J. Irwin, 'Normal form methods for complicated periodic systems', Part. Accel **24**, 91 (1989)

[5] E. Forest, 'Analytical computation of the smear', SSC-95 (1987), and J. Bengtsson and J. Irwin, 'Analytical calculations fo smear and tune shift', SSC-232 (1990)

[6] J. Irwin, 'A multi-kick factorization algorithm for nonlinear maps', SSC--228 (1989).

[7] D. Abell, 'Analytical properties and Cremona approximation of transfer maps for Hamiltonian systems', U. of Maryland physics department, PhD thesis (1995)

[8] G. Roy, 'Analysis of the optics of the final focus test beam using Lie algebra based techniques', PhD Thesis, SLAC Report-397 (1992]

[9] F. Zimmermann, R. Helm and J. Irwin, Optimization of the NLC final focus system, SLAC-PUB-6791.

[10] J. Irwin, N. Walker and M. Woodley, 'Using Lie algebra methods to analytically study non-perturbative effects in beam lines', SLAC-PUB-5920 (1992)

[11] N. Walker, J. Irwin, and M. Woodley, 'Third order corrections to the SLC final focus, SLAC-PUB-6205 (1993)

[12] Y. T. Yan, J. Irwin, and T. Chen, 'Resonance basis maps and nPB tracking for dynamic aperture studies', SLAC-PUB-7179, June 96

[13] Y. T. Yan, Y. Cai, D. Ritson, T. Chen, and J. Irwin, 'Swamp plots for dynamic aperture studies of PEPII lattices', SLAC-PUB-6876