ROBINSON-TYPE CRITERIA FOR BEAM AND RF CAVITY WITH DELAYED, VOLTAGE-PROPORTIONAL FEEDBACK

Shane Koscielniak, TRIUMF, 4004 Wesbrook Mall, Vancouver B.C., Canada

Abstract

We generalize the Robinson stability criteria, for a charged particle beam interacting with the radio-frequency cavity resonator that is responsible for accelerating it, to the case that the resonator is equipped with a delayed voltage-proportional feedback.

1 INTRODUCTION

Robinson[1] gave criteria for the stability of a charged particle beam interacting with the RF cavity resonator that is responsible for accelerating that beam. A widely adopted procedure[3] for high current beams, to avoid the power-limited instability, is to reduce the apparent cavity impedance by feedback. Inevitably, the feedback is delayed; and this introduces exponential terms into the system characteristic equation. We give a general, exact, analytic procedure for determining whether there are poles/zeros in the right-half complex plane (RHCP); and apply the method to find analogues of the Robinson stability criteria when the resonator is equipped with a delayed feedback. Of course, one should determine the stability of the delayed-feedback-resonator alone, before analysis with beam. The beam instability criteria are explained, in physical terms, in Ref. [6] using the theory of Sacherer[2]. This article is a precis of a more pedagogic exposition given in References [5, 6].

1.1 Nyquist stability criterion

Let $s = \sigma + j\omega$ be the Laplace frequency and $j = \sqrt{-1}$. If the system transfer function $F(s)$ has any poles fall in the RHCP, then the system is unstable. We adopt the usual convention of the complex plane that positive rotations are counter-clockwise. Nyquist realized that the difference in number of poles and zeros in the RHCP is equal to the number of counter-clockwise encirclements of the origin by the locus of the function $F(s)$ as s varies along a counter-clockwise semi-circular contour in the RHCP with the imaginary axis as diameter. Hence, Nyquist reduced the stability analysis to merely counting up loops about the origin. Given that we are searching for poles, it often simplest to decompose F into a numerator and denominator. Then, for the stability analysis, we investigate under what conditions the denominator has zeros in the RHCP. F shall now stand for the denominator of the transfer function.

1.2 Analytic stability criterion

We want a criterion that is easy to apply, and preferably algebraic rather than geometric – so that curve sketching is avoided. We divide $F(s)$ into a real part $\mathcal{A} = \Re[F]$ and an imaginary part $\mathcal{B} = \Im[F]$. It is clear that we want the locus of F to rotate counter-clockwise, and so we consider the angular rotation rate $d\,Arg(F)/d\omega = [\mathcal{A}\mathcal{B}' - \mathcal{A}'\mathcal{B}]/[\mathcal{A}^2 + \mathcal{B}^2]$ which is positive for positive rotations. Fortunately, we do not need to consider this quantity for all values of ω.

1.2.1 Criterion for poles and no zeros

In order to encircle the origin, the curve traced by $F(\omega)$ has to move through the four quadrants of the complex plane; and to do this there must be places where either $\mathcal{A}$ or $\mathcal{B}$ changes sign. Hence we are interested in the roots ω_A of $\mathcal{A} = 0$ and the roots ω_B of $\mathcal{B} = 0$. We state conditions so that F has only poles but no zeros, that is criteria that ensure the counter-clockwise encirclement of the origin.

$$\mathcal{A}(\omega_B) \times \mathcal{B}'(\omega_B) \;>\; 0 \quad \text{and/or} \tag{1}$$

$$\mathcal{B}(\omega_A) \times \mathcal{A}'(\omega_A) \;<\; 0\,. \tag{2}$$

These conditions are sketched in the R.H.S. of Figure 1.

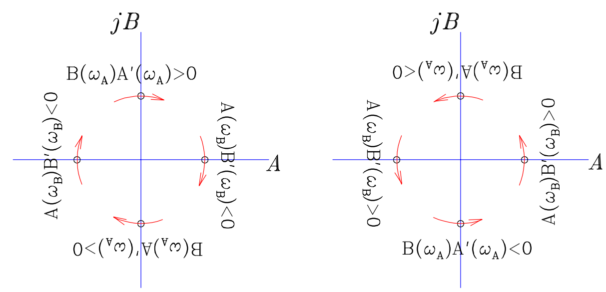

Figure 1: Geometric interpretation of stability criteria.

The stability criterion $\mathcal{A}\mathcal{B}' - \mathcal{B}\mathcal{A}' > 0$ does not have to be satisfied at all the roots ω_A and ω_B; but it must be satisfied at those which can cause encirclement of the origin. If there is no root ω_A between the nearest neighbour roots $\omega_B(n)$ and $\omega_B(n+1)$, then there is no encirclement. Hence, we should only apply the stability criteria where the roots *alternate*; that is where $\omega_B(n+1) > \omega_A > \omega_B(n)$ for condition (1), or where $\omega_A(n+1) > \omega_B > \omega_A(n)$ for condition (2).

1.3 The problem of delay

By delay, we mean that the value of some quantity at time t is related to the value of some other quantity at an earlier time $t - T$, where T is the delay interval. It is simple to show that the Laplace transform of $F(t - T)$ is $\exp(-sT)F(s)$ provided that $F(t) = 0$ for $t < T$. The Nyquist criterion is applicable to the exponential function because it is the limit of a polynomial: $\exp(-sT) = (1 - sT/N)^N$ as $N \to \infty$ where N is an integer. Suppose we set $T = 0$ and find a finite set of roots. Roots which satisfy $\omega T \ll 1$ will not be much shifted when we allow τ

to become finite; we call them the 'LF roots.' When $T > 0$, there will also be an infinite set of HF roots; and these are usually periodic or approximately so; of these, usually only a very small subset can cause encirclement of the origin.

2 RESONATOR WITH DELAYED FEEDBACK

In the neighbourhood of resonance, a cavity behaves like a simple LCR parallel resonator. Let the original resonance angular frequency be Ω, the shunt resistance R_s and the time constant be $\tau_c = 1/\alpha = 2Q/\Omega$ where Q is the quality factor. Let us suppose the resonator is equipped with a delayed feedback of gain A/R_s. We suppose the delay interval is τ. Let dots placed above a variable denote derivatives with respect to time, t. The voltage, V, and driving current, I, obey the equation:

$$\ddot{V} + 2\alpha[\dot{V}(t) + A\dot{V}(t-\tau)] + \Omega^2 V = 2\alpha R_s \dot{I} \; . \quad (3)$$

For simplicity, suppose that $|A| \gg 1$ and that A does not cause any phase-shift. Also, for brevity, let us write $2\alpha A = B$ and $2\alpha R = C$. We introduce a dimensionless 'time' $u = \Omega t$, and dimensionless variables $b = B/\Omega$, $T = \Omega\tau$ and $c = C/\Omega$. Let primes denote derivatives with respect to u. We form the Laplace transform to find:

$$[s^2 + bse^{-sT} + 1]V(s) = csI(s) \; . \quad (4)$$

The eigen-values of the equation obtained by setting $I(s) = 0$ are the natural free-oscillation frequencies of the system.

2.1 *Criteria for stability*

We set $s = j\omega$ and form $F = \mathcal{A} + j\mathcal{B}$. It is easiest to locate the roots ω_B of $\mathcal{B} = 0$. We need to find one low frequency root and an infinite, periodic set of high frequency roots.

2.1.1 Low frequency root

If $\omega_B = 0$ is the only root for which $\mathcal{A}(\omega_B) > 0$ then we find the condition $\mathcal{A}\mathcal{B}' > 0$ implies $b > 0$. However, if $T > \pi/2$ then there will be two or more roots ω_B for which $\mathcal{A}(\omega_B) > 0$; and so $b > 0$ is not an essential condition. In fact, b must change sign periodically as T increases, and the maximum allowed gain b passes through zero at $\omega_n^2 = 1$ or $T = (2n+1)(\pi/2)$.

2.1.2 High frequency roots

The other roots of $\mathcal{B}$ occur at $\omega_n = (2n+1)\pi/(2T)$ with $n = 0, 1, 2, \ldots$ a positive integer. We find the condition:

$$\mathcal{A}\mathcal{B}' = [1 - \omega_n^2 + b\,\omega_n(-1)^n][-b\,\omega_n T(-1)^n] > 0 \; . \quad (5)$$

This need not be satisfied for all n. However, it must be satisfied for the two adjacent n which cause $\mathcal{A}$ to change sign. The exact conditions depend on the delay T.

The inequality $\Omega\tau = T \leq \pi/2$ is a special case: a sufficient condition for stability is $\mathcal{A}(\omega_0)\mathcal{B}'(\omega_0) > 0$ with $\omega_0 = \pi/(2T)$; which implies:

$$0 < B\tau < \left[(\pi/2) - (2/\pi)(\Omega\tau)^2\right] \; . \quad (6)$$

The case $T = \Omega\tau > \pi/2$ is more complicated. Suppose that $T \approx m\pi$ where m is the nearest integer. We establish the quantities T_{crt} and b_{max}:

$$\begin{aligned} T_{crt} &= m\pi\sqrt{1 - 1/(4m^2)} && \rightarrow m\pi && (7) \\ (-1)^m b_{max} &= \pi/T_{crt} && \rightarrow 1/m \quad \text{for large } m \; . && (8) \end{aligned}$$

T_{crt} is a 'critical' value of the delay. Below T_{crt}, the limiting gain is the solution of $\mathcal{A}_{m-1} = 0$. Above T_{crt}, b is the solution of $\mathcal{A}_m = 0$. Exactly at T_{crt} the limiting stable gain is equal to b_{max}. Now we may state the extremal gain conditions.

$$(2m-1)\pi/2 \leq T \leq T_{crt} \Rightarrow b = (-1)^m[1-\omega_{m-1}^2]/\omega_{m-1} \; . \quad (9)$$

$$T_{crt} \leq T \leq (2m+1)\pi/2 \Rightarrow b = (-1)^m[\omega_m^2 - 1]/\omega_m \; . \quad (10)$$

The gain stability boundary is sketched in figure 2. If the gain b (with appropriate sign) is smaller than given in conditions (9, 10) then all natural oscillations are self damped.

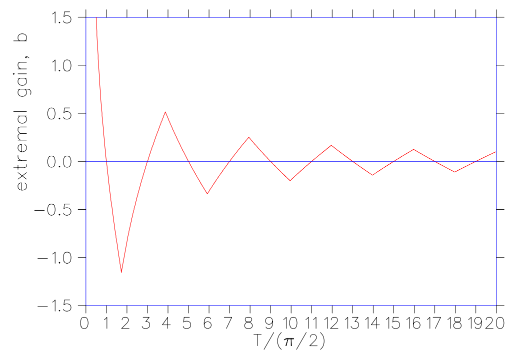

Figure 2: Maximum values of gain, b compatible with stability versus delay T.

2.2 *Radio-frequency system*

If the resonator appears as a real load when driven at Ω, then one must take the cases $T = \Omega\tau = m\pi > T_{crt}$. Hence the gain limit is given by:

$$(-1)^m \times A = Q/m = (\tau_c/\tau)(\pi/2) \; . \quad (11)$$

Other choices are possible: take the condition $\omega_{\rm rf}\tau = m\pi$ where $\omega_{\rm rf} \neq \Omega$ is the desired drive radio-frequency. Below transition, $\omega_{\rm rf} < \Omega$ and so $T > m\pi$. Let us introduce the detuning angle by the definition:

$$\tan\Psi = (\Omega^2 - \omega^2)/(2\alpha\omega) = Q(\Omega-\omega)(\Omega+\omega)/(\Omega\omega) \, . \quad (12)$$

Let $T = m\pi + (\Omega - \omega_{\rm rf})\tau \approx m\pi + (\tau/\tau_c)\tan\Psi$ and substitute in (10). $\Psi > 0$ because $\omega_{\rm rf} < \Omega$. In the limit of large m, the extremal gain is approximately

$$A(-1)^m \approx (Q/m)\,[1 - (2/\pi)(\tau/\tau_c)\tan\Psi] \; , \quad (13)$$

3 ANALOGUES OF THE ROBINSON CRITERIA

We consider small perturbations about the steady state, and develop the analysis in terms of the transfer functions for phase and amplitude modulations of the carrier frequency. Let us suppose the gain, A, has been chosen consistent with the delay τ. The drive current is the sum of a generator component and the fundamental beam current component $\mathbf{I}_b^0 = jI_b^0 e^{j\Phi_b}$. It is customary[4] to define $I_V^0 = V^0/R$ and introduce the ratio $Y_b = I_b^0/I_V^0$. Let $w = \omega_{\rm rf}\tau + \theta$ where θ is a fixed arbitrary phase-shift in the feedback. We define the synchrotron frequency Ω_s *sans* the usual trigonometric factor and also define $\omega_s^2 = \Omega_s^2 \cos\Phi_b$. Using the beam response equations of References[4, 5], for small dipole oscillations of the bunch about the steady state phase, Φ_b, the system matrix is given by:

$$\begin{bmatrix} m_{11} & m_{12} & -Y_b\cos\Phi_b \\ m_{21} & m_{22} & +Y_b\sin\Phi_b \\ -\Omega_s^2\sin\Phi_b & -\Omega_s^2\cos\Phi_b & s^2+\Omega_s^2\cos\Phi_b \end{bmatrix} \begin{bmatrix} a_v \\ \phi_v \\ \phi_b \end{bmatrix} = \ldots$$

$$m_{11} = +m_{22} \;=\; 1 + s\tau_c + Ae^{-s\tau}\cos w \quad (14)$$

$$m_{12} = -m_{21} \;=\; Ae^{-s\tau}\sin w + \tan\Psi\ . \quad (15)$$

The natural frequencies are obtained by setting the determinant equal to zero, leading to $c_4s^4+c_3s^3+c_2s^2+c_1s+c_0 = 0$ where the polynomial coefficients contain exponential terms. We set $s = +j\omega$ to find $F = \mathcal{A} + j\mathcal{B}$.

3.1 *Low frequency roots*

$\mathcal{B}$ has three low frequency roots.

3.1.1 Root at $\omega = 0$

The quantity $\mathcal{A}(0)\mathcal{B}'(0)$ is is easiest to interpret when $w = m\pi$ and m is even, in which case the criterion:

$$[(1+A)^2 + \tan^2\Psi - Y_b\tan\Psi] \times [\tau_c - A\tau] > 0\ . \quad (16)$$

We know $A\tau < \tau_c$; and so it follows that $Y_b\tan\Psi < (1+A)^2 + \tan^2\Psi$ is a necessary condition for stability. This is formally identical with Robinson's 'power limited' stability criterion; and this was anticipated because delay cannot change the nature of a d.c. instability – for d.c. signals, an arbitrarily long delay does not change the signal.

3.1.2 Roots at $\pm\omega_s$

ω_s is an exact root of $\mathcal{B}$ when $w = m\pi$. Let us evaluate

$$\mathcal{A}(\omega_s)\mathcal{B}'(\omega_s) \;=\; 4\Omega_s^2 Y_b[1 + (-1)^m A\cos(\omega_s\tau)]\,\omega_s \times[\omega_s\tau_c - (-1)^m A\sin(\omega_s\tau)]\tan\Psi\ . \quad (17)$$

This quantity can become negative, indicating instability, in a variety of ways. Suppose m is even. Without feedback, the real part of the cavity impedance is positive (i.e. dissipative) at synchrotron upper and lower sidebands, and the difference is proportional to $\omega_s\tau_c \times \tan\Psi$. With feedback, the real part of the impedance at the sidebands becomes proportional to $[\omega_s\tau_c - (-1)^m A\sin(\omega_s\tau)] \times \tan\Psi$. However, from the stability of the cavity without beam we know

$$\tau_c > A\tau \geq A\sin(\omega_s\tau)/\omega_s\ . \quad (18)$$

Hence, from (17) we conclude the stability conditions:

$$\tan\Psi \;>\; 0\,, \quad (19)$$

$$1 + A\cos(\omega_s\tau) \;>\; 0\ . \quad (20)$$

The inequality (19) is the first Robinson criterion and it tells us to detune the cavity in the correct sense: $\Omega > \omega_{\rm rf}$ when below transition energy.

Suppose feedback phasing is adjusted so that the real part of the at the drive frequency is positive. For sufficiently long delay and high synchrotron frequency, one finds that at the upper and lower synchrotron sidebands of the drive frequency, the real part of the impedance looks like a *negative* resistance. Condition (20) determines if this situation occurs; from which we conclude $\omega_s\tau < \pi/2$ or $\omega_s < \omega_0$.

3.2 *High frequency roots*

To simplify matters we shall consider the case of very large gain, that is $|A| \gg 1$. The $\omega_n = (2n+1)\pi/(2\tau)$ where n is an integer are exact roots of $\mathcal{B} = 0$ if $w = m\pi$. When $\omega = \omega_n$ and $w = m\pi$, then

$$\mathcal{A} \;=\; -\Omega_s^2 Y_b\tan\Psi + (\omega_n^2 - \Omega_s^2\cos\Phi_b) \times \times[A^2 - 2(-1)^{m+n}A\tau_c\omega_n + (\tau_c\omega_n)^2 - \tan^2\Psi] \quad (21)$$

$$\mathcal{B}' \;=\; 2A\tau(-A + (-1)^{m+n}\tau_c\omega_n)(\omega_n^2 - \Omega_s^2\cos\Phi_b) \quad (22)$$

To simplify, let m be even. The transfer functions are only valid for small modulation frequencies and so we set $n = 0$. Now $\omega_0 > \omega_s$ and $A < \omega_0\tau_c$ from Eqn. (11), and so:

$$\left(\frac{Q\pi}{\Omega\tau} - A\right)^2 > \tan^2\Psi + \frac{(\Omega_s\tau)^2 Y_b\tan\Psi}{(\pi/2)^2 - (\Omega_s\tau)^2\cos\Phi_b}\ . \quad (23)$$

This condition implies that the maximum, stable feedback gain is reduced under conditions of heavy beam loading. The condition is only accurate under the condition of long delay: $\Omega\tau \gg \pi/2$. If relation (23) is violated, then a coherent oscillation occurs at the frequency $\omega_0 = \pi/(2\tau)$ because the dipole mode frequency is shifted away from the nominal synchrotron frequency by the very large reactance at the sidebands of the carrier that occurs if $|\tau_c\omega_0 - A| \approx \tan\Psi$. Essentially, the reactive impedance raises the coherent frequency to a point high enough that it can oscillate in synchronism with a spontaneous high frequency oscillation of the resonator-with-feedback. Despite the fact that the feedback is in-phase at the carrier, at this sideband frequency ω_0 the feedback is in quadrature so making the effective impedance look very reactive.

Reference[5] generalizes all results to the case of arbitrary w and θ and A.

4 REFERENCES

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