PARTICLE MOTION IN THE STABLE REGION NEAR THE EDGE OF A LINEAR SUM RESONANCE STOPBAND

G. Parzen*, Brookhaven National Laboratory, Upton, NY 11973 USA

Abstract

This paper studies the particle motion when the tune is in the stable region close to the edge of linear sum resonance stopband. Results are found for the tune and the beta functions. Results are also found for the two solutions of the equations of motion. The results found are shown to be also valid for small accelerators where the large accelerator approximation may not be used.

1 INTRODUCTION

This paper studies the motion of a particle whose tune is near an edge of a linear sum resonance stopband. It is assumed that the tune is not near any other linear resonance, and the motion is dominated by the linear sum resonance. It is assumed that the linear sum resonance is being driven by a skew quadrupole field perturbation. When the unperturbed tune ν_{x0}, ν_{y0} is close to the resonance line $\nu_x + \nu_y = q$, q being an integer, the particle motion can be unstable. Results are found for the tune and the beta functions when the unperturbed tune is in the stable region but close to an edge of the stopband. Results are also found for the two solutions of the equations of motion. All the results found are shown to be also valid for small accelerators where the large accelerator approximation may not be used. See [14] for more details.

2 RESULTS WHEN THE TUNE IS INSIDE THE STOPBAND

It will be assumed that in the absence of the perturbing fields, the tune of the particle is given by ν_{x0}, ν_{y0}, the x and y motions are uncoupled, and that the motion is stable when ν_{x0}, ν_{y0} is close to the line $\nu_{x0} + \nu_{y0} = q$, where q is an integer. It is assumed that a perturbing field is then added which is given by the skew quadrupole field

$$\begin{aligned} \Delta B_x &= -B_0 a_1\, x \\ \Delta B_y &= B_0 a_1\, y \end{aligned} \qquad (2\text{-}1)$$

a_1 is the skew quadrupole multipole and $a_1 = a_1(s)$. B_0 is some standard field, usually the field in the main dipoles of the lattice.

The coupled equations of motion can be written as

$$\begin{aligned} \frac{d^2}{d\theta_x^2}\eta_x &+ \nu_{x0}^2 \eta_x = b_x\, \eta_y \\ \frac{d^2}{d\theta_y^2}\eta_y &+ \nu_{y0}^2 \eta_y = b_y\, \eta_x \end{aligned} \qquad (2\text{-}2)$$

$$\begin{aligned} b_x &= -\nu_{x0}^2 \beta_x (\beta_x\beta_y)^{\frac{1}{2}} a_1/\rho \\ b_y &= -\nu_{y0}^2 \beta_y (\beta_x\beta_y)^{\frac{1}{2}} a_1/\rho \end{aligned}$$

One can assume that η_x has the form

$$\begin{aligned} \eta_x &= A_s \exp(i\nu_{xs}\theta_x) + \sum_{r\neq s} A_r \exp(i\nu_{xr}\theta_x) \\ \nu_{xr} &= \nu_{xs} + n, \quad n \text{ an integer}, n \neq 0 \end{aligned} \qquad (2\text{-}3a)$$

where for small enough a_1, $A_r \ll A_s$ and $\nu_{xs} \to \nu_{x0}$ for $a_1 \to 0$. For the corresponding form for η_y one might assume for η_y

$$\begin{aligned} \eta_y &= \sum_r B_r \exp(i\nu_{yr}\theta_y) \\ \nu_{yr} &= \nu_{xs} + n \end{aligned} \qquad (2\text{-}3b)$$

where $B_r \ll A_s$ for small enough a_1.

It will be seen below, that the solution assumed for η_y Eq. (2-3b) is valid if one is not near the sum resonance $\nu_x + \nu_y = q$, q being an integer. When ν_{x0}, ν_{y0} are close to the sum resonance $\nu_x + \nu_y = q$, then one of the B_r will become as large as A_s and this is the B_r for which $\nu_{yr} = \nu_{xs} - q$. This is shown below. Thus, one assumes for η_y the solution with the form

$$\begin{aligned} \eta_y &= B_{\overline{s}} \exp(i\nu_{y\overline{s}}\theta_y) + \sum_{r\neq \overline{s}} B_r \exp(i\nu_{yr}\theta_y) \\ \nu_{y\overline{s}} &= \nu_{xs} - q \\ \nu_{yr} &= \nu_{xs} + n, \quad n \neq -q \end{aligned} \qquad (2\text{-}3c)$$

Here $B_r \ll A_s$ but $B_{\overline{s}} \simeq A_s$. It is being assumed that ν_{x0}, ν_{y0} are not close to any other resonance other than $\nu_x + \nu_y = q$.

Putting this assumed form for η_x, η_y into the differential equations Eq. (2-2), and assuming for the initial guess

$$\begin{aligned} \eta_x &= A_s \exp(i\nu_x\theta_x) \\ \eta_y &= B_{\overline{s}} \exp(i\nu_{y\overline{s}}\theta_y) \\ \nu_{y\overline{s}} &= \nu_{xs} - q = -(q - \nu_{xs}) \end{aligned} \qquad (2\text{-}4)$$

one finds that, see [14] for details,

$$\begin{aligned} (\nu_{xs}^2 - \nu_{x0}^2)(\nu_{y\overline{s}}^2 - \nu_{y0}^2) &= 4\nu_{x0}\nu_{y0}|\Delta\nu_x|^2 \\ \Delta\nu_x &= \frac{1}{4\pi}\int_0^L ds (\beta_x\beta_y)^{\frac{1}{2}} (a_1/\rho) \\ &\exp[-i(\nu_{x0}\theta_x + (q-\nu_{x0})\theta_y)] \end{aligned} \qquad (2\text{-}5)$$

To solve Eq. (2-5) one puts

$$\nu_{xs} = \nu_{xsR} - ig_x$$

* Work performed under the auspices of the U.S. Department of Energy.

where ν_{xsR} and g_x are both real, which gives the equation

$$(\nu_{xsR} - ig_x - \nu_{x0})(q - \nu_{xsR} + ig_x - \nu_{y0}) = |\Delta\nu_x|^2 \quad (2\text{-}6)$$

Eq. (2-6) then gives

$$\begin{aligned} g_x^2 &+ [\frac{1}{2}(q - \nu_{x0} - \nu_{y0})]^2 = |\Delta\nu_x|^2 \\ g_x &= \pm\left\{|\Delta\nu_x|^2 - [\frac{1}{2}(q - \nu_{x0} - \nu_{y0})]^2\right\}^{1/2} \end{aligned} \quad (2\text{-}7)$$

Results can be found for η_x and η_y which are correct to first order in the perturbation and when ν_{x0}, ν_{y0} is inside the stopband or in the stable region near an edge of the stopband. η_x, η_y are given by Eqs. (2-3). For the ν_x mode

$$\begin{aligned} \eta_x &= A_s \exp(i\nu_{xs}\theta_x) + \sum_{r \neq s} A_r \exp(i\nu_{xr}\theta_x) \\ \eta_y &= B_{\bar{s}} \exp(i\nu_{y\bar{s}}\theta_y) + \sum_{r \neq s} B_r \exp(i\nu_{yr}\theta_y) \\ \nu_{y\bar{s}} &= \nu_{xs} - q \\ \nu_{yr} &= \nu_{xs} + n, \;\; n \neq -q \\ \nu_{xr} &= \nu_{xs} + n, \;\; n \neq 0 \end{aligned} \quad (2\text{-}8)$$

Results and details for η_x, η_y are given in [14].

3 THE TUNE NEAR THE EDGE OF A STOPBAND

In this section, a result will be found for the tune in the stable region outside the stopband but close to an edge of the stopband. It will be shown that close to an edge of the stopband the tune of the ν_x mode is given by

$$\begin{aligned} |\nu_x - \frac{1}{2}(\nu_{x0} + q - \nu_{y0})| &= \{\epsilon_x |\Delta\nu_x|\}^{\frac{1}{2}} \\ \epsilon_x &= |q \pm 2|\Delta\nu_x| - \nu_{x0} - \nu_{y0}| \end{aligned} \quad (3\text{-}1)$$

ν_x is the tune of the ν_x mode, ϵ is the distance from ν_{x0}, ν_{y0} to the edge of the stopband. In the $\pm$, the $+$ sign is for the upper edge, and the $-$ sign for the lower edge. When ν_{x0}, ν_{y0} reaches the edge of the stopband, then $\epsilon = 0$, and $\nu_x = \frac{1}{2}(\nu_{x0} + q - \nu_{y0})$ is the real part of the tune inside the stopband.

Eq. (3-1) shows that near the stopband edge, ν_x varies rapidly with ϵ_x. As one reaches the edge of the stopband, ϵ_x goes to zero and $d\nu_x/d\epsilon_x$ becomes infinite like $\epsilon_x^{-\frac{1}{2}}$.

To find ν_x in the stable region outside the stopband, where $|q - \nu_{x0} - \nu_{y0}| > 2|\Delta\nu_x|$, one goes back to the derivation given in section 2 for ν_x inside the stopband, starting with

$$(\nu_x - \nu_{x0})(q - \nu_x - \nu_{y0}) = |\Delta\nu_x|^2 \quad (3\text{-}2)$$

Because of the condition that ν_x is outside the stopband or

$$|q - \nu_{x0} - \nu_{y0}| > 2|\Delta\nu_x| \quad (3\text{-}3)$$

one sees that one must have $g_x = 0$.

Let us assume that we start with ν_{x0}, ν_{y0} below the lower stopband edge and let ν_{x0}, ν_{y0} approach the lower stopband edge. The equation of the lower stopband edge is given by

$$q - \nu_{x0} - \nu_{y0} = 2|\Delta\nu_x| \quad (3\text{-}4)$$

when ν_{x0}, ν_{y0} arrive on the lower stopband edge, then ν_x will arrive at the value $\nu_x = \frac{1}{2}(\nu_{x0} + q - \nu_{y0})$. Thus below the stopband edge one can write

$$\nu_x = \frac{1}{2}(\nu_{x0} + q - \nu_{y0}) - \delta_x \quad (3\text{-}5)$$

where $\delta_x \to 0$ when ν_{x0}, ν_{y0} arrive at the stopband edge. We then find

$$\begin{aligned} \nu_x - \nu_{x0} &= \frac{1}{2}(q - \nu_{x0} - \nu_{y0}) - \delta_x \\ q - \nu_x - \nu_{y0} &= \frac{1}{2}(q - \nu_{x0} - \nu_{y0}) + \delta_x \end{aligned} \quad (3\text{-}6)$$

and Eq. (3-2) becomes

$$\begin{aligned} &[\frac{1}{2}(q - \nu_{x0} - \nu_{y0})]^2 - \delta_x^2 = |\Delta\nu_x|^2 \\ &\delta_x = \left\{[\frac{1}{2}(q - \nu_{x0} - \nu_{y0})]^2 - |\Delta\nu_x|^2\right\}^{\frac{1}{2}} \end{aligned} \quad (3\text{-}7)$$

Eq. (3-7) gives ν_x in the stable region near the stopband. It can be put in another form that indicates the dependence on the distance from ν_{x0}, ν_{y0} to the stopband edge.

Below the stopband, one writes

$$\epsilon_x = q - 2|\Delta\nu_x| - \nu_{x0} - \nu_{y0} \quad (3\text{-}8)$$

where ϵ_x indicates the distance from ν_{x0}, ν_{y0} to the stopband edge which is given by Eq. (3-4). When ν_{x0}, ν_{y0} is on the stopband edge and $\nu_{x0} + \nu_{y0} = q - 2|\Delta\nu_x|$ then $\epsilon_x = 0$.

Using Eq. (3-8) to replace $q - \nu_{x0} - \nu_{y0}$ by $\epsilon_x + 2|\Delta\nu_x|$ in Eq. (3-7) one finds

$$\delta_x = \{\epsilon_x(|\Delta\nu_x| + \epsilon_x/4)\}^{\frac{1}{2}} \quad (3\text{-}9)$$

Eq. (3-9) can then be written so as to hold both above and below the stopband to give

$$\begin{aligned} \left|\nu_x - \frac{1}{2}(\nu_{x0} + q - \nu_{y0})\right| &= \{\epsilon_x(|\Delta\nu_x| + \epsilon_x/4)\}^{\frac{1}{2}} \\ \epsilon_x &= |q \pm 2|\Delta\nu_x| - \nu_{x0} - \nu_{y0}| \end{aligned} \quad (3\text{-}10)$$

where ϵ_x is the distance from ν_{x0}, ν_{y0} to the stopband edge. One uses the $+$ sign for the upper stopband edge and the $-$ sign for the lower edge.

Close to the stopband edge, where $\epsilon_x \ll |\Delta\nu_x|$ then Eq. (3-10) gives the result

$$\left|\nu_x - \frac{1}{2}(\nu_{x0} + q - \nu_{y0})\right| = \{\epsilon_x\,|\Delta\nu_x|\}^{1/2} \quad (3\text{-}11)$$

Equations (3-10) and (3-11) give the tune of the ν_x mode, ν_x, near the stopband edge. The result for the tune of the ν_y

mode, ν_y, may be found by making the substitution $\nu_x \rightarrow \nu_y, \nu_{x0} \rightarrow \nu_{y0}, \nu_{y0} \rightarrow \nu_{x0}, |\Delta\nu_x| \rightarrow |\Delta\nu_y|$.

If one varies the unperturbed tune, ν_{x0}, ν_{y0}, so that the tune approaches the edge of the stopband, the tune on the stopband edge depends on the value of ν_{x0}, ν_{y0} when the unperturbed tune arrives at the stopband edge. The stopband edges are given by the two lines

$$\nu_{x0} + \nu_{y0} = q \pm 2|\Delta\nu|$$

where it is assumed that $|\Delta\nu_x| = |\Delta\nu_y| = |\Delta\nu|$ and the $+$ sign is for the upper edge and the $-$ sign for the lower edge.

The tune of the ν_x mode at the stopband edge is then given by

$$\begin{aligned} \nu_x &= \frac{1}{2}(\nu_{x0} + q - \nu_{y0}) \\ \nu_x &= \nu_{x0} \pm |\Delta\nu| \end{aligned} \qquad (3\text{-}12)$$

where the $+$ sign is for the lower edge and the $-$ sign for the upper edge.

The tune of the ν_y mode at the stopband edge is given by

$$\nu_y = \nu_{y0} \pm |\Delta\nu|$$

One may note, that at the stopband edge

$$\begin{aligned} \nu_x + \nu_y &= \nu_{x0} + \nu_{y0} \pm 2|\Delta\nu| \\ \nu_x + \nu_y &= q \end{aligned} \qquad (3\text{-}13)$$

and the ν_x, ν_y lies on the resonance line.

Eqs. (3-6) and (3-7) can also be rewritten as, for the ν_x mode and below the resonance line,

$$\begin{aligned} \nu_x &= \nu_{x0} + 0.5D\left\{1 - [(\frac{2\Delta\nu}{D})^2]^{\frac{1}{2}}\right\} \\ D &= q - \nu_{x0} - \nu_{y0} \\ \Delta\nu &= \Delta\nu_x \simeq \Delta\nu_y \end{aligned} \qquad (3\text{-}14)$$

Results for the beta functions near the edge of the stopband are given in [14].

4 COMMENTS ON THE RESULTS

Others have worked on this subject and there is an overlap between the contents of this paper and their work. These previous papers [4–13] give results for the stopband width and for the growth rate inside the stopband.

The results in this paper include the following:

1. Results for the tune in the stable region near an edge of the stopband. The results show that as ν_{x0}, ν_{y0} approach the edge of the stopband, the tunes of the two normal modes ν_x and ν_y begin to change rapidly and when ν_{x0}, ν_{y0} reach the stopband edge then ν_x and ν_y lie on the resonance line $\nu_x + \nu_y = q$. These final values of ν_x, ν_y, when ν_{x0}, ν_{y0} reach the stopband edge, are approached like $\epsilon^{\frac{1}{2}}$, where ϵ is the distance from ν_{x0}, ν_{y0} to the stopband edge.
2. Results for the beta functions of the normal modes, β_x, β_y, in the stable region near the edge of a stopband. The results show that β_x, β_y do not become infinite when ν_{x0}, ν_{y0} approach the stopband edge, unless ν_{x0}, ν_{y0} are near the half integer resonances $v_x = m/2$, or $v_y = n/2$, m and n being integers.
3. Results for the 2 solutions of the equations of motion in the stable region near a stopband edge and in the unstable region.
4. The above results hold also for small accelerators, where the exact equations of motion have to be used and the large accelerator approximation is not valid. For small accelerators, one needs the restriction that the perturbing field gradients do not shift the closed orbit.

5 REFERENCES

[1] G. Parzen, Linear orbit parameters for the exact equations of motion, BNL Report, BNL–60090 (1994).

[2] G. Parzen, Particle motion in the stable region near the edge of a linear half–integer resonance, BNL report, BNL–62036 (1995).

[3] D. Edwards and L. Teng, Parameterization of linear coupled motion in periodic systems, Proc. 1973 IEEE PAC, p. 885 (1973).

[4] P.A. Sturrock, Static and dynamic electron optics, Cambridge Univ. Press, London (1955).

[5] R. Hagedom, Stability and amplitude ranges of two–dimensional non–linear oscillations with a periodic hamiltonian, CERN 57-1 (1957).

[6] A. Schoch, Theory of linear and non-linear perturbations of betatron oscillations in an alternating gradient synchrotron, CERN 57-21 (1957).

[7] E.D. Courant and H.S. Snyder, Theory of the alternating gradient synchrotron, Ann, Phys. 3, 1 (1958).

[8] A.A. Kolomensky and A.N. Lebedev, Theory of cyclic accelerators, North Holland Publishing Co. (1966).

[9] G. Ripken, Untersuchugen zur Strahlführung und Stabsilitat der Teilchen bewegung, DESY R1-7014 (1970).

[10] W.P. Lysenko, Nonlinear betatron oscillations, Particle Accelerators 5, 1 (1973).

[11] G. Guignard, The general theory of all sum and difference resonances in three–dimensional magnetic field in a synchrotron, CERN 76-06 (1976).

[12] G. Guignard, A general treatment of resonances in accelerators, CERN 78-11 (1978).

[13] H. Wiedenmann, Particle Accelerator Physics II, Springer–Verlag, New York (1995).

[14] G. Parzen, Particle motion in the stable region near the edge of a linear sum resonance, BNL Report, BNL-62456 (1995).