PARTICLE MOTION INSIDE AND NEAR A LINEAR HALF-INTEGER STOPBAND

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Abstract

This paper studies the motion of a particle whose tune is inside and near a linear half-integer stopband. Results are found for the tune and beta functions in the stable region close to an edge of the stopband. It is shown that the eigenvalues and the eigenfunctions of the transfer matrix are real inside the stopband. All the results found are also valid for small accelerators where the large accelerator approximation is not used.

1 INTRODUCTION

Inside a linear-half integer stopband the particle motion can be unstable and grow exponentially. The eigenvalues and eigenfunctions of the transfer matrix are shown to be real inside the stopband. In the stable region near an edge of the stopband, the tune varies rapidly and the beta function becomes infinite as the unperturbed tune approaches the edge of the stopband. Results are found for the tune and beta function in the stable region near an edge of the stopband. It is found that the beta function becomes infinite inversely as the square root of the distance of the unperturbed tune from the edge of the stopband.

2 RESULTS WHEN THE TUNE IS IN THE STOPBAND

It will be assumed that in the absence of the perturbing fields, the tune of the particle is ν_0 and that the motion is stable when ν_0 is close to $q/2$, where q is an integer.

It is assumed that a perturbing field is present which is given on the median plane by

$$\Delta B_y = -G(s)x \tag{2-1}$$

$G(s)$ is periodic in s and contains the field harmonics that can excite the stopband around $\nu_0 = q/2$.

Introducing η defined by

$$\eta = x/\beta^{1/2}, \tag{2-2}$$

where β is the beta function of the unperturbed field, the equation of motion can be written as

$$\begin{aligned} \frac{d^2\eta}{d\theta^2} &+ \nu_0^2\eta = f \\ f &= \nu_0^2\beta^{3/2}\Delta B_y/B\rho \\ f &= -\nu_0^2\beta^2 G\eta/B\rho \\ B\rho &= pc/e,\ d\theta = ds/\nu_0\beta \end{aligned} \tag{2-3}$$

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The final results found below are valid for small accelerators that require the use of the exact linearized equations. See [7] for details.

Eq. (2-3) can be written as

$$\begin{aligned} \frac{d^2\eta}{d\theta^2} + \nu_0^2 &= -2\nu_0 b(\theta)\eta \\ b(\theta) &= \frac{1}{2}\nu_0\beta^2 G/B\rho \end{aligned} \tag{2-4}$$

Because $b(\theta)$ is periodic a solution for η will have the form

$$\eta = \exp(i\nu_s\theta)h(\theta) \tag{2-5}$$

where $h(\theta)$ is periodic. It is assumed that the tune ν_0 will change to ν_s because of the perturbing field. Thus η can be assumed to have the form

$$\begin{aligned} \eta &= A_s\exp(i\nu_s\theta) + \sum_{r\neq s} A_r\exp(i\nu_r\theta) \\ \nu_r &= \nu_s + n \end{aligned} \tag{2-6}$$

where n is some integer but $n \neq 0$. For a zero perturbing field the solution for η is $\eta = A\exp(i\nu_0\theta)$. Thus for small perturbing fields it can be assumed that

$$\begin{aligned} \nu_s &\simeq \nu_0 \\ A_r &\ll A_s \text{ for } r \neq s \end{aligned} \tag{2-7}$$

Putting Eq. (2-6) into Eq. (2-4), one obtains a set of equations for the A_r

$$\begin{aligned} (\nu_r^2 - \nu_0^2)A_r &= 2\nu_0\sum_{\overline{r}} b_{r\overline{r}}A_{\overline{r}} \\ b_{r\overline{r}} &= \frac{1}{2\pi}\int_0^{2\pi} d\theta\, b(\theta)\exp(-i\nu_r\theta + i\nu_{\overline{r}}\theta) \\ \nu_r &= \nu_s + n \end{aligned} \tag{2-8}$$

It is assumed that ν_0 is near $\nu_0 = q/2$, q being an integer. The stopband is defined as the range of ν_0 for which the tune, ν_s, in the presence of the perturbation given by Eq. (2-2) has a non-zero imaginary part. One can write ν_s as

$$\nu_s = \nu_{sR} - ig \tag{2-9}$$

It will be shown that inside the stopband, where $g \neq 0$, then

$$\nu_{sR} = q/2 \tag{2-10}$$

This may be shown as follows. Let μ be the phase shift for a period, and $\mu = 2\pi\nu_s$ where the period has been assumed

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to be one turn or 2π in θ. Let T be the transfer matrix for one period. Then one has

$$\cos\mu = \frac{1}{2}(T_{11} + T_{22}), \tag{2-11}$$

and one sees that $\cos\mu$ is real even inside the stopband since the T_{ij} are real. One also has

$$\begin{aligned}\cos\mu &= \cos 2\pi(\nu_{sR} - ig) = \cos(2\pi\nu_{sR})\cosh(2\pi g) \\ &+ i\sin(2\pi\nu_s R)\sin(2\pi g)\end{aligned} \tag{2-12}$$

In order for $\cos\mu$ to be real, one has to have either $g = 0$ or, if $g \neq 0$, $2\pi\nu_{sR} = n\pi$, where n is some integer. Thus, inside the stopband where $g \neq 0$,

$$\nu_{sR} = n/2 \tag{2-13}$$

In order to have continuity when the perturbation goes to zero, one has $n = q$ and

$$\nu_{sR} = q/2 \tag{2-14}$$

Now let us return to the problem of computing the growth factor, g, using perturbation theory. One sees from Eq. (2-8) that when $\nu_0 \simeq q/2$, then one of the A_r becomes comparable to A_s, and this is the A_r for which $\nu_r = \nu_s - q$, $\nu_r \simeq -q/2$. Thus in the above iterative procedure for finding η, one will assume for the initial guess for η,

$$\begin{aligned}\eta &= A_s\exp(i\nu_s\theta) + A_{\overline{s}}\exp(i\nu_{\overline{s}}\theta) \\ \nu_{\overline{s}} &= \nu_s - q\end{aligned} \tag{2-15}$$

From Eq. (2-8) one finds

$$\begin{aligned}(\nu_r^2 - \nu_o^2)A_r &= 2\nu_0 b_{rs}A_s + 2\nu_0 b_{r\overline{s}}A_{\overline{s}} \\ \nu_r = \nu_s + n \quad &\text{or} \quad \nu_r = \nu_{\overline{s}} + n = \nu_s - q + n\end{aligned} \tag{2-16}$$

For $r = s$ and $r = \overline{s}$ one obtains 2 equations for A_s and $A_{\overline{s}}$

$$\begin{aligned}(\nu_s^2 - \nu_0^2)A_s &= 2\nu_0 b_{s\overline{s}}A_{\overline{s}} \\ (\nu_{\overline{s}}^2 - \nu_0^2)A_{\overline{s}} &= 2\nu_0 b_{\overline{s}s}A_s \\ \nu_{\overline{s}} &= \nu_s - q\end{aligned} \tag{2-17}$$

In Eq. (2-17), it has been assumed, for simplicity sake, that $b_{ss} = 0$. This can be accomplished by redefining ν_0 to be $\nu_0 + b_0$.

In order for Eqs. (2-17) to have a solution, one must have

$$\begin{aligned}(\nu_s^2 - \nu_0^2)(\nu_{\overline{s}}^2 - \nu_0^2) &= 4\nu_0^2|b_{s\overline{s}}|^2 \\ \nu_{\overline{s}} &= \nu_s - q\end{aligned} \tag{2-18}$$

Eq. (2-18) determines ν_s. If one writes $\nu_s = \nu_{sR} - ig$, one finds

$$\left[(\nu_{sR} - ig)^2 - \nu_0^2\right]\left[(\nu_{sR} - q - ig)^2\right] = 4\nu_0^2|\Delta\nu|^2 \tag{2-19}$$

where $\Delta\nu = b_{s\overline{s}}$

$$\Delta\nu = \frac{1}{4\pi}\int_0^L ds\,\beta\exp(-iq\theta)G/B\rho \tag{2-20}$$

$$\nu_{sR} = q/2 \tag{2-21}$$

$$\begin{aligned}(q/2 - \nu_0)^2 + g^2 &= |\Delta\nu|^2 \\ g &= \pm\left\{|\Delta\nu|^2 - (q/2 - \nu_0)^2\right\}^{1/2}\end{aligned} \tag{2-22}$$

3 TUNE NEAR THE EDGE OF A STOPBAND

In this section, a result will be found for the tune in the stable region outside the stopband but close to one of the edges of the stopband. It will be shown that close to the edge of a stopband,

$$|\nu - q/2| = \{2|\Delta\nu|\,|\nu_0 - \nu_e|\}^{1/2} \tag{3-1}$$

ν is the tune in the presence of the gradient perturbation, ν_e is the edge of the stopband, $\nu_e = q/2 \pm |\Delta\nu|$. $|\Delta\nu|$ is the half-width of the stopband. Eq. (3-1) shows that when ν_0 is close to an edge of the stopband, ν_e, ν varies rapidly with ν_0, and the slope of the ν vs. ν_0 curve is vertical at $\nu_0 = \nu_e$.

To find ν in the stable region outside the stopband, where $\nu_0 - q/2| > |\Delta\nu|$, one goes back to the derivation given for ν inside the stopband, starting with Eq. (2-19). Eq. (2-22) shows that for $|\nu_0 - q/2| > |\Delta\nu|$ the only acceptable solution is $g = 0$, and Eq. (2-19) can be written as

$$(\nu - \nu_0)(|\nu - q| - \nu_0) = |\Delta\nu|^2 \tag{3-2}$$

where we have put $\nu_s = \nu$.

Assuming that ν_0 is just below the stopband edge $\nu_e = q/2 - |\Delta\nu|$, put $\nu_0 = \nu_e - \epsilon$ and $\nu = q/2 - \delta$ into Eq. (3-2), where ϵ and δ both approach zero as ν_0 approaches the stopband edge. We find

$$\delta = \{\epsilon(\epsilon + 2|\Delta\nu|)\}^{1/2} \tag{3-3}$$

The top edge of the stopband can be treated in the same way and both results can be combined into the one result

$$\begin{aligned}|\nu - q/2| &= \{|\nu_0 - \nu_e|(|\nu_0 - \nu_e| + 2|\Delta\nu|)\}^{1/2} \\ \nu_e &= q/2 \pm |\Delta\nu|\end{aligned} \tag{3-4}$$

Very close to the stopband edge, $|\nu_0 - \nu_e| \ll |\Delta\nu|$, one finds

$$|\nu - q/2| = \{2|\Delta\nu||\nu_0 - \nu_e|\}^{1/2} \tag{3-5}$$

Thus, as ν_0 approaches a stopband edge, ν approaches $q/2$, and $d\nu/d\nu_0$ become infinite like $1/|\nu_0 - \nu_e|^{1/2}$.

A result can be found for the beta function in the stable region outside the stopband, but close to one of the edges of the stopband. It can be shown that close to edge of a stopband (see [7] for details)

$$[(\beta - \beta_0)/\beta_0]_{\max} = [2|\Delta\nu|/|\nu_0 - \nu_e|]^{1/2} \tag{3-6}$$

ν_e is the edge of the stopband, $\nu_e = q/2 \pm |\Delta\nu|$. $|\Delta\nu|$ is the half-width of the stopband. ν_0, β_0 are the unperturbed tune and beta function. Eq. (3-6) shows that when ν_0 approaches the edge of the stopband, $(\beta - \beta_0)/\beta_0$ becomes infinite like $1/|\nu_0 - \nu_e|^{1/2}$.

4 COMMENTS ON THE RESULTS

Others have worked on this subject and there is some overlap between the contents of this paper and their work. P.A.

Sturrock[2] obtained results for the stopband width and the growth parameter g at the center of the stopband. E.D. Courant and H. Snyder[3] obtained the result for the width of the stopband. H. Bruck[4] showed that the solutions of the equation of motion at the edge of the stopband are stable. He also states that the real part of ν is constant at $q/2$ across the stopband without giving a proof of this. A.A. Kolomensky and A.N. Lebedev[5] obtained results for the stopband width and the growth parameter g at the center of the stopband. H. Wiedemann[6] obtained the result for the width of the stopband using a method similar to that used by Courant and Snyder.

The new results in this paper include the following. The result given for the tune ν near the edge of the stopband, ν_e, $|\nu - q/2| = [2|\Delta\nu||\nu_e - \nu_0|]^{1/2}$. The result given for the beta function near the edge of the stopband, $[(\beta - \beta_0)/\beta_0]_{\max} = [2|\Delta\nu|/|\nu_e - \nu_0|]^{1/2}$. The proof given showing that the real part of ν is constant over the stopband at $q/2$ does not depend on perturbation theory, and the result follows from the symplectic properties. The result that all the results found in this paper will also hold for a small accelerator where the large accelerator approximation is not used. The result given for the solutions of the equations of motion when ν_0 is inside the stopband, and the proof that the eigenfunctions and eigenvalues of the transfer matrix are real inside the stopband. See [7] for more details.

5 REFERENCES

[1] G. Parzen, Linear orbit parameters for the exact equations of motion, BNL Report, BNL-60090 (1994).

[2] P.A. Sturrock, Static and dynamic electron optics, Cambridge Univ. Press, London (1955).

[3] E.D. Courant and H.S. Snyder, Theory of the alternating gradient synchrotron, Annals. of Physics, Vol. 3, p. 1 (1958).

[4] H. Bruck, Circular particle accelerators, (1965), English translation Los Alamos Report, LA-TR-72-10 Rev.

[5] A.A. Kolomensky and A.N. Lebedev, Theory of cyclic accelerators, North Holland Publishing Co., Amsterdam (1966).

[6] H. Wiedemann, Particle accelerator physics, Springer-Verlag, Berlin (1993).

[7] G. Parzen, Particle motion inside and near a linear half-integer stopband, BNL Report BNL-62036 (1995).