ORBIT DYNAMICS FOR UNSTABLE LINEAR MOTION

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Abstract

A treatment is given of the orbit dynamics for linear unstable motion that allows for the zeros in the beta function and makes no assumptions about the realness of the betatron and phase functions. The phase shift per turn is shown to be related to the beta function and the number of zeros the beta function goes through per turn. The solutions of the equations of motion are found in terms of the beta function.

1 INTRODUCTION

In the case of linear unstable motion, the beta function can be zero at some points in the lattice. Because of the zeros in the beta function, and other assumptions often made about the realness of the beta function and phase function, the usual treatment given for stable motion does not carry over to the case of unstable motion. A treatment is given below, that allows for the zeros in the beta functions and does not make assumptions about the realness of the betatron and phase functions.

It will be shown that the solutions of the equations of motion can be written in the form

$$\begin{aligned} x &= \beta^{\frac{1}{2}} \exp(\pm\psi) \\ \psi &= P\int_{s_0}^{s} \frac{ds}{\beta} + i\frac{\pi}{2}N_z \end{aligned} \qquad (1\text{-}1)$$

N_z is the number of times $\beta(s)$ goes through zero between s_0 and s. P indicates the principle value of the integral. The solutions of the equations of motion can also be written as

$$x = \exp[\pm\mu s/L] f(s) \qquad (1\text{-}2)$$

where $f(s)$ is periodic and L is the length of one turn. It will be shown that for unstable motion

$$\begin{aligned} \mu &= 2\pi(g + iq/2) \\ q &= \frac{1}{2}N_z \\ g &= \frac{P}{2\pi}\int_0^L \frac{ds}{\beta} \end{aligned} \qquad (1\text{-}3)$$

where N_z is the number of zeros the beta function goes through in one turn. P indicates the principle value of the integral.

Often, the case of unstable linear motion is found when a gradient perturbation is applied to a lattice whose unperturbed ν-value is close to $q/2$, q being some integer. In this case, perturbation theory will show [1] that the solutions have the form given by Eq. (1-1) where $q/2$ is the half integer close to the unperturbed to the ν-value. In the general case, where the unstable motion cannot be viewed as due to a perturbing gradient then the value of q is given by $\frac{1}{2}N_z$ where N_z is the number of zeros in the beta function in one turn.

It will also be shown that near a zero of the beta function at $s = s_1$, ψ will become infinite and the dominant term is ψ is given by

$$\psi \sim \pm\frac{1}{2}\log(s - s_1) \qquad (1\text{-}4)$$

See [4] for more details.

2 THE DEFINITION OF THE BETA FUNCTION

The linear parameters can be defined in terms of the elements of the one period transfer matrix. The 2×2 transfer matrix, M, is defined by

$$\begin{aligned} x(s) &= M(s, s_0)x(s_0) \\ x &= \begin{pmatrix} x \\ p_x \end{pmatrix} \end{aligned} \qquad (2\text{-}1)$$

The one period transfer matrix is defined by

$$\hat{M}(s) = M(s + L, s) \qquad (2\text{-}2)$$

where the lattice is assumed to be periodic with the period L. The matrix M is assumed to be symplectic

$$\begin{aligned} M\overline{M} &= I \\ \overline{M} &= \widetilde{S}\widetilde{M}\, S \\ S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \qquad (2\text{-}3)$$

$\widetilde{S}$ is the transpose of S. Also $|M| = 1$ where $|M|$ is the determinant of M. One can show that $\hat{M}(s)$ and $\hat{M}(s_0)$ are related by

$$\hat{M}(s) = M(s, s_0)\hat{M}(s_0)M(s_0, s) \qquad (2\text{-}4)$$

It follows from Eq. (2-4) that $\hat{M}_{11} + \hat{M}_{22}$, the trace of $\hat{M}$, is independent of s. For unstable motion it is assumed that $|\hat{M}_{11} + \hat{M}_{22}| > 2$. This may be shown to lead to unstable exponentially growing motion.

One can now introduce the constant parameter μ defined by

$$\cosh\mu = \frac{1}{2}\left(\hat{M}_{11} + \hat{M}_{22}\right) \qquad (2\text{-}5)$$

If $\hat{M}_{11} + \hat{M}_{22}$ is positive, then μ will be real. However if $\hat{M}_{11} + \hat{M}_{22}$ is negative then μ has to have the imaginary

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part $iq\pi$ where q is an odd integer. In general, one can write

$$\begin{aligned} \mu &= \mu_R + iq\pi \\ \cosh \mu_R &= \frac{1}{2}\left|\hat{M}_{11} + \hat{M}_{22}\right| \end{aligned} \tag{2-6}$$

where q is an even integer if $\hat{M}_{11} + \hat{M}_{22}$ is positive, and q is an odd integer when $\hat{M}_{11} + \hat{M}_{22}$ is negative.

μ is related to the eigenvalues of $\hat{M}$, λ_1 and λ_2, where $\lambda_1 + \lambda_2 = \hat{M}_{11} + \hat{M}_{22}$ and $\lambda_1\lambda_2 = 1$ from $|\hat{M} - \lambda I| = 0$. It follows from Eq. (2-5) that

$$\begin{aligned} \lambda_1 &= \exp(\mu) \\ \lambda_2 &= \exp(-\mu) \end{aligned} \tag{2-7}$$

One can define the linear parameters, β, α, γ, using the elements of the one period transfer matrix. If one uses the form of the transfer matrix often used [2] for stable motion the linear parameters will be imaginary for unstable motion. To make the linear parameters real, they will be defined here in terms of the one period transfer matrix as

$$\hat{M} = \begin{bmatrix} \cosh\mu + \alpha\sinh\mu & \beta\sinh\mu \\ \gamma\sinh\mu & \cosh\mu - \alpha\sinh\mu \end{bmatrix} \tag{2-8}$$

$$\beta\gamma = 1 - \alpha^2$$

β, α, γ are then given in terms of $\hat{M}_{ij}$ as

$$\begin{aligned} \beta &= (-1)^q \hat{M}_{12} / \sinh\mu_R \\ \alpha &= (-1)^q (\hat{M}_{11} - \hat{M}_{22}) / 2\sinh\mu_R \\ \gamma &= (1 - \alpha^2)/\beta \end{aligned} \tag{2-9}$$

Eq. (2-6) does not specify the sign of μ_R. One can define the sign of μ_R to be always positive. Then β, α, γ can then be computed from the $\hat{M}_{ij}$ using Eq. (2-9). It will be seen later that the sign of $\beta(s)$ can change within a period, and $\beta(s)$ can be zero at certain values of s for unstable motion.

2.1 Differential Equations for β, α, γ

It is assumed that the linearized equations of motion can be written as

$$\begin{aligned} \frac{dx}{ds} &= A_{11}x + A_{12}p_x \\ \frac{dp_x}{ds} &= A_{21}x + A_{22}p_x \\ A_{11} + A_{22} &= 0 \end{aligned} \tag{2-10}$$

In the large accelerator approximation, $A_{11} = A_{22} = 0$ and $A_{12} = 1$. one can show that

$$\frac{d\hat{M}}{ds} = A\,\hat{M} - \hat{M}\,A. \tag{2-11}$$

A is the 2×2 matrix whose elements are the A_{ij} of Eq. (2-10). Replacing $\hat{M}$, using Eq. (2-8), in Eq. (2-13) gives the result

$$\begin{aligned} \frac{d\beta}{ds} &= 2A_{11}\beta - 2A_{12}\alpha \\ \frac{d\alpha}{ds} &= -A_{21}\beta + A_{12}\gamma \\ \frac{d\gamma}{ds} &= 2A_{21}\alpha - 2A_{11}\gamma \end{aligned} \tag{2-12}$$

2.2 Differential Equation for β

In this section, the differential equation for β will be obtained without making any assumptions about the form of the solutions of the equations of motion. For the sake of simplicity, the derivation will be given for the large accelerator case which assumes $A_{11} = A_{22} = 0$ and $A_{12} = 1$.

Introducing b, where $\beta = b^2$, one can then find (see [4] for details)

$$\begin{aligned} \frac{d^2b}{ds^2} + Kb + \frac{1}{b^3} &= 0 \\ b &= \beta^{\frac{1}{2}}. \end{aligned} \tag{2-13}$$

3 SOLUTIONS OF THE EQUATIONS OF MOTION AND THE BETA FUNCTION

For stable motion, the role of the beta function in the solutions of the equations of motion is well known. A similar result will be found here for unstable motion. The treatment usually given for stable motion, does not carry over to unstable motion because of the assumptions usually made about the realness of the betatron and phase functions, and the absence of zeros in the beta function.

Let us write the solutions of the equations of motion as

$$\begin{aligned} x &= b\exp(\psi) \\ b &= \beta^{\frac{1}{2}} \end{aligned} \tag{3-1}$$

where β and b have been defined by Eq. (2-8). Then b has been shown to obey, see Eq. (2-13),

$$\begin{aligned} \frac{d^2b}{ds^2} &+ Kb + \frac{1}{b^3} = 0 \\ K &= -A_{21} \end{aligned} \tag{3-2}$$

x then obeys the equations

$$\frac{d^2x}{ds^2} + Kx = 0 \tag{3-3}$$

Putting the form of x assumed in Eq. (3-1) into Eq. (3-3), and using Eq. (3-2) for b one gets

$$\frac{d^2\psi}{ds^2} + \frac{2}{b}\frac{db}{ds}\frac{d\psi}{ds} + \left(\frac{d\psi}{ds}\right)^2 - \frac{1}{b^4} = 0 \tag{3-4}$$

Putting $f = d\psi/ds$ one gets

$$\frac{df}{ds} + \frac{2}{b}\frac{db}{ds}f + f^2 - \frac{1}{b^4} = 0 \tag{3-5}$$

The solutions of Eq. (3-5) are

$$f = \pm(1/b^2) = \pm 1/\beta. \tag{3-6}$$

Thus

$$\psi = \pm \int_{s_0}^{s} \frac{ds}{\beta} \tag{3-7}$$

and the two solutions of the equations of motion are

$$x = \beta^{\frac{1}{2}} \exp\left(\pm \int_{s_0}^{s} \frac{ds}{\beta}\right) \tag{3-8}$$

One may note that in deriving Eq. (2-8) no assumption was made about the realness of β or ψ. However, there is a problem with the result for unstable motion, as in the case of unstable motion $\beta(s)$ will go through zero. To evaluate the integral when $\beta(s)$ has zeros, Eq. (3-2) will be replaced by

$$\psi = \lim_{\epsilon \to 0} \int_{s_0}^{s} \frac{ds}{\beta - i\epsilon} \tag{3-9}$$

where ϵ is a positive small quantity. It can be shown that Eq. (3-9) gives (see section 4)

$$\psi = P \int_{s_0}^{ds} \frac{ds}{\beta} + \sum_{s_n} \frac{i\pi}{|\beta'(s_n)|} \tag{3-10}$$

where s_n are the locations of the zeros of $\beta(s)$ from s_0 to s. P represents the principle part of the integral.

One can also show that $\beta'(s) = \pm 2$ at the zeros of $\beta(s)$. Since $\beta\gamma = \alpha^2 - 1$, then $\alpha = \pm 1$ when $\beta = 0$. Since $\beta' = -2\alpha$, $\beta' = \mp 2$ when $\beta = 0$. One can now write Eq. (3-10) as

$$\psi = P \int_{s_0}^{s} \frac{ds}{\beta} + \frac{i\pi}{2} N_z \tag{3-11}$$

where N_z is the number zeros in $\beta(s)$ in s_0 to s.

One may notice that the imaginary part of ψ has on an unusual dependence on s. It is constant in between zeros of $\beta(s)$ and jumps by $\pi/2$ at each zero of $\beta(s)$. One can use Eq. (3-11) to find the change in ψ over one turn, $\psi(s+L) - \psi(s)$, and find

$$\psi(s+L) - \psi(s) = P \int_{s}^{s+L} \frac{ds}{\beta} + iq\pi \tag{3-12}$$

where $2q$ is the number of zeros in $\beta(s)$ in one turn, and L is the length of one turn. For simplicity, it is being assumed that the period L is one turn. Since $\beta(s)$ is a periodic function, the number of zeros of $\beta(s)$ in one turn has to be even. If one defines the tune as the imaginary part of $\psi(s+L) - \psi(s)$ divided by 2π, then one has

$$\text{tune} = q/2. \tag{3-13}$$

Eq. (4-13) shows the connection between the tune and the number of zeros in the beta function in one turn. The real part of $\psi(s_0+L) - \psi(s_0)$ gives the exponential growth in one turn. If one defines the exponential growth factor, g, to be the real part of $\psi(s+L) - \psi(s)$ divided by 2π

$$g = \frac{P}{2\pi} \int_{s}^{s+L} \frac{ds}{\beta} \tag{3-14}$$

See [4] for details.

4 PHASE FUNCTION RESULTS WHEN β HAS ZEROS

In this section, the result for the phase function, ψ, given by Eq. (3-10) will be derived. Also, the behavior of ψ when s is near the zeros of $\beta(s)$ will be studied. First, let us consider the case where

$$\psi = \lim_{\epsilon \to 0} \int_{s_0}^{s} \frac{ds}{\beta - i\epsilon} \tag{4-1}$$

$\epsilon > 0$, and one assumes there is only one zero for $\beta(s)$ at $s = s_1$ between $s = s_0$ to $s = s$. Then, one can write

$$\psi = P \int_{s_0}^{s} \frac{ds}{\beta} + \int_{s_1-\delta}^{s_1+\delta} \frac{ds}{\beta - i\epsilon} \tag{4-2}$$

where $\delta \to 0$ but $\delta \gg \epsilon$. P stands for the principle part of the integral. Near s_1 one can write $\beta = \beta'(s_1)(s - s_1) + ...$ and find

$$\begin{aligned}
\int_{s_1-\delta}^{s_1+\delta} \frac{ds}{\beta - i\epsilon} &= \int_{s_1-\delta}^{s_1+\delta} \frac{ds}{\beta'(s_1)(s-s_1) - i\epsilon} \\
&= \frac{1}{\beta'(s_1)} \int_{-\delta}^{\delta} d\overline{s} \frac{(\overline{s} + i\overline{\epsilon})}{\overline{s}^2 + \overline{\epsilon}^2}, \ \overline{s} = s - s_1, \ \overline{\epsilon} = \epsilon/\beta'(s_1) \\
&= \frac{1}{\beta'(s_1)} \frac{i\overline{\epsilon}}{|\overline{\epsilon}|} \pi \\
&= \frac{1}{|\beta'(s_1)|} i\pi
\end{aligned} \tag{4-3}$$

If there are many zeros between s_0 to s at $s = s_n$ one then finds

$$\psi = P \int_{s_0}^{s} \frac{ds}{\beta} + \sum_{s_n} \frac{i\pi}{|\beta'(s_n)|} \tag{4-4}$$

It can be shown that near a zero of $\beta(s)$, like $s = s_1$, ψ becomes infinite like

$$\psi \sim \pm \frac{1}{2} \log(s - s_1) \tag{4-5}$$

see [4] for more details and results.

5 REFERENCES

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