EIGENMODES OF MICROWAVE CAVITIES CONTAINING HIGH-LOSS DIELECTRIC MATERIALS

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Abstract

The numerical determination of the electromagnetic field eigenmodes of a microwave cavity containing regions of high-loss dielectric material is of technological importance to many areas, including high power microwave generation, particle accelerator design and microwave sintering of ceramic materials. This problem has proved problematic to numerical techniques [1], causing poor convergence and long computation times when highly lossy materials are present. The Jacobi-Davidson algorithm [2] applied to this problem is shown to be capable of extracting a set of eigenmodes, even for the cases of degenerate eigenvalues and low ohmic-Q cavity modes. Details of the theory and numerical solution using 2-dimensional (planar circuit) and 3-dimensional electromagnetic operators will be presented.

1 INTRODUCTION

Many scientific and engineering problems require the determination of a selected number of the eigenvalues, and possibly eigenvectors, of a large system of linear equations. Typical examples arise in electromagnetics when a field is represented by discrete values and the determining equation is discretised in some manner, for example by finite difference, finite element or finite integration methods. The ease with which such problems may be solved depends on the size of the problem, but also is strongly dependent upon the distribution of the eigenvalues and whether they are real or complex.

Methods for eigenvalue and eigenvector computation for large, linear systems of equations by iterative methods frequently require that the matrices be self-adjoint (Hermitian). Such methods can often be applied with some success for systems having eigenvalues with relatively small imaginary parts, however the theoretical basis for convergence is not well defined and poor convergence often results. Many basic properties of the solutions of self-adjoint eigensystems do not hold in the general case, and more rigourous techniques must be used if convergence is to be ensured.

2 EIGENVALUE FORMULATION

Two electromagnetic eigenproblems will be considered here. Firstly, a 2-dimensional planar circuit model will be used to provide realistic test examples of moderate size, for which solutions may be readily visualised. Secondly, a full 3-dimensional model is described and examples given to show the effectiveness of the Jacobi-Davidson method on larger problems.

2.1 2D planar circuit model

Simple electromagnetic planar circuits considered here are those for which an exact two-dimensional representation can be found in terms of a single field component. The class includes many structures of practical use, for example rectangular waveguides with H-plane bends and junctions, which may contain columnar structures of metal or dielectric. Such circuits have no structure in the third dimension other than parallel bounding planes of metal, and possess only modes having an electric field perpendicular to the plane.

For such a circuit having no structure in the y direction, the following eigenvalue problem may be derived

$$\left\{\nabla_\perp^2 + \omega^2 \epsilon(\mathbf{r}_\perp)\mu_0\right\} E_y(\mathbf{r}_\perp) = 0$$

The cavity frequencies, ω, and mode structures, $E_y(\mathbf{r}_\perp)$, may be obtained as solutions of this eigensystem. Metallic wall boundary conditions in the x–z plane may be included in the numerical implementation of the differential operator. The transverse Laplacian operator is self-adjoint, and therefore if the permittivity and permeability functions are real then the eigensystem is symmetric, and the spectrum will contain only real eigenvalues.

2.2 3D complex cavity model

A 3-dimensional electromagnetic operator was derived using the finite integration method [3]. The eigenvalue form derived from Maxwell's equations is (in normalised units) the vector equation [1]

$$\mathrm{curl}\ \mu^{-1}\mathrm{curl}\ \mathbf{E} - \mathrm{grad}\ \mathrm{div}\ \epsilon\mathbf{E} = \omega^2\epsilon\mathbf{E}$$

where the discretised components of $\mathbf{E}$ form the eigenvector, and ω^2 is the eigenvalue of the numerical implementation of the operator. The divergence term is included to remove the multiply degenerate *static* solutions with eigenvalue zero which result from the freedom of E to include a component corresponding to the gradient of an arbitrary scalar field.

The finite integration method defines field components using the Yee cell arrangement, shown in figure 1, for which the differential operators take a particularly simple form, and takes account of the permittivity and permeability of each

cell.

2.3 LOSSY MATERIALS

Regions of complex permittivity, representing Ohmic loss in these materials, make the discretised eigensystems non-Hermitian, and therefore critically change the characteris-

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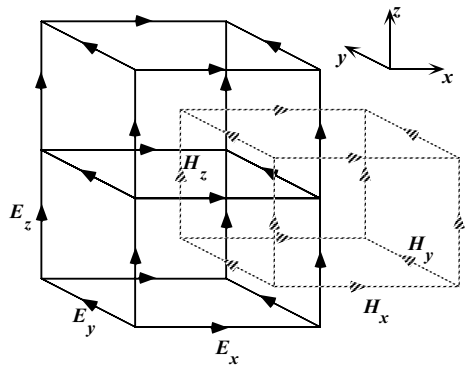

Figure 1: Yee cell structure of 3D discretisation

tics of the solution. The solution for non-Hermitian systems may often be obtained using methods derived for Hermitian systems, provided that the system deviates only slightly from a Hermitian problem. Here, this corresponds to small values of the loss tangent of the dielectric materials, and solutions for such cases frequently converge using Hermitian methods. If highly lossy materials are to be modelled successfully, a more rigourous approach is necessary.

3 NON-HERMITIAN EIGENSYSTEMS

A generalised linear matrix eigensystem can be written in the form

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{B}\mathbf{x} \tag{1}$$

The column vector $\mathbf{x}$ satisfying (1) is an eigenvector of the system $\{\mathbf{A},\mathbf{B}\}$, with corresponding eigenvalue, λ. For matrices which are not Hermitian, it is possible to formulate a second, related eigenproblem as follows.

$$\overline{\mathbf{y}}\mathbf{A} = \lambda\overline{\mathbf{y}}\mathbf{B} \;\text{ or }\; \overline{\mathbf{A}}\mathbf{y} = \overline{\mathbf{B}}\mathbf{y}\overline{\lambda} \tag{2}$$

where $\overline{\mathbf{y}}$ is a row vector. The vector $\overline{\mathbf{y}}$ satisfying (2) is a *left*-eigenvector of the system $\{\mathbf{A},\mathbf{B}\}$, with corresponding eigenvalue, λ. For a given generalised eigensystem described by $\{\mathbf{A},\mathbf{B}\}$, pairs of left- and right-eigenvectors have equal eigenvalues, though the eigenvectors are not necessarily equal.

The left and right eigenvectors possess a mutual orthogonality relationship, such that each left (right) eigenvector with corresponding eigenvalue λ is orthogonal to each right (left) eigenvector with a different corresponding eigenvalue.

4 THE JACOBI-DAVIDSON METHOD

The Jacobi-Davidson technique is applicable to eigenproblems of the generalised form (1). Here, it has been extended to operate with independent left and right subspaces, so that the left and right eigenvectors may be represented effectively for non-Hermitian problems. In the standard method, a single subspace is used to span the space containing the desired left *and* right eigenvectors, for which the theory is well described by Sleijpen *et al.* [4]. At each step, the subspace is updated by obtaining an estimate of the correction to each approximate eigenvector using Jacobi's orthogonal component correction (JOCC) method derived from the residual error. For systems which are not self-adjoint, the left and right eigenvectors corresponding to an eigenvalue will in the general case not be parallel, and the subspaces spanned by the eigenvectors corresponding to a given set of eigenvalues will be different. Therefore separate subspaces should be maintained and updated for the solution of non-Hermitian systems. This has been suggested in connection with the original Davidson method for use with non-normal matrices [5].

The modified Jacobi-Davidson method for non-Hermitian systems operates as follows. The large eigensystem is projected at each iteration into the current stored left and right subspaces, and the small projected problem solved exactly to obtain estimates of the solution through a Ritz procedure. Separate subspace corrections are obtained using JOCC steps on the left and right eigenvector residuals, and incorporated in the subspaces. Both corrections may be obtained from a single procedure, and little computational overhead is involved in obtaining the second correction as a by-product of calculating the first.

5 EXAMPLE GEOMETRIES

To test the effectiveness of the method in solving for eigenfrequencies and field eigenmodes of cavities, results from a number of test examples will be given. The ability of an algorithm to converge can depend on the characteristics of the structure of the cavity. For example,

- symmetry of the structure can lead to degeneracy of eigenvalues
- materials with high loss-tangent can give rise to non-real eigenvalues in the spectrum
- the number of unknowns in the solution may influence the convergence

The test examples are intended to include sufficient complexity with regard to these criteria in order to ascertain the effectiveness of the eigenvalue solver for electromagnetic problems.

5.1 2D geometries

Two-dimensional (planar circuit) problems are simple to visualise, and permit accurate solutions to be obtained for relatively complex structures with a moderate number of unknowns, typically 10^4 to 10^5. Two examples are given here.

5.1.1 Lossy structure with degenerate modes

The first example consists of a 2mm$\times$2mm square column of lossy dielectric material located centrally in a 20mm$\times$20mm square cavity, discretised on a 100×100

cell grid. The dielectric constant of the block was assumed to be $\epsilon_r = 10 - 2i$.

The following table shows the real and imaginary components of the complex eigenfrequency and the cavity-Q for each of the first eight cavity eigenmodes. Degenerate eigenvalues converged to the limit of numerical round-off error.

Mode	Complex Freq./GHz		Q
1	6.2145185	0.48687	6.38
2/3	13.8931731	0.807264	8.60
4	16.6253632	0.370592	22.43
5/6	19.8161824	0.636433	15.56
7	20.5588852	0.485044	21.19
8	23.1571167	0.394130	29.37

5.1.2 Loaded cavity test structure

Figure 2 shows an eigenmode of a cavity in a cylindrical geometry, loaded with a lossy ceramic ring. The permittivity of the ceramic was ϵ_r=12.24, with loss tangent 0.3. Though convergence time increased for this structure, the eigenmodes were successfully identified. Localised modes in the ceramic having Q<4 were also observed.

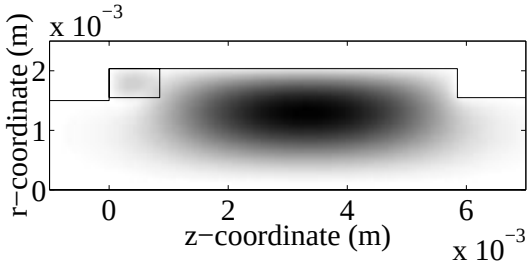

Figure 2: Gyroklystron cavity mode at 93.37GHz

The following table shows good agreement for two similar cavities between eigenmodes determined by (a) this calculation and (b) a scattering matrix calculation:

Mode:		TE_{011}	TE_{012}	TE_{012}	TE_{012}
(a)	f	93.377	103.095	92.867	102.517
	Q	174.198	51.941	187.077	53.654
(b)	f	93.491	103.396	92.929	102.736
	Q	168.767	46.864	169.635	47.045

These values are also in excellent agreement with experimental cold-test data.

5.2 3D geometries

Three-dimensional geometries increase the problem complexity in a number of ways

- vector field increases the number of unknowns in the solution by a factor of three
- many more cells are necessary to fill a 3-dimensional region while maintaining solution accuracy
- more coupling terms arise in the matrix operator

5.2.1 Cubic cavity with high degeneracy

A perfectly cubic cavity with a symmetric discretisation provides an excellent test for separation of degenerate eigenvalues. This method successfully identified the six-fold (real) eigenvalue degeneracies of this structure, correct to round-off error (approx 10^{-14}).

5.2.2 Lossy structure with degenerate modes

The final test example follows Schmitt *et al.* [1] and consists of a cavity 20mm×20mm×10mm with a lossy dielectric block 7mm×7mm×8mm located centrally on the square cavity floor. The dielectric constant of the block was taken as $\epsilon_r = 10 - 2i$. Calculations were performed using a 40×40×10 cell mesh with approximately 4.8×10^4 unknowns, slightly coarser than the reference example. The following agreement was obtained.

Mode	Freq./GHz		Q	Freq./GHz [1]	
1	6.1384	0.2810	10.92	6.161	0.278
2/3	9.0865	0.7790	5.83	9.091	0.780
4/5	11.316	0.7583	7.46	11.39	0.759
6	11.416	1.0386	5.49	11.42	1.104
7	13.250	1.1593	5.71	13.25	1.161
8	13.618	0.8746	7.78	13.66	0.870
9/10	13.667	0.8750	7.81	13.78	0.860

The method used here is, however, better applicable to such non-Hermitian systems, and converges uniformly even for loss tangents greater than one.

6 CONCLUSION

The Jacobi-Davidson method for eigenvalue determination has been modified to operate with separate left and right subspaces, and shown to successfully identify eigenmodes of complex cavities containing highly absorbing materials. The method does not transform the matrix operator directly, and so may be used if the operator and its adjoint are available in functional form. Additionally, there is no requirement for estimation of extreme eigenvalues, and the solution may be seeded for iterative refinement. The method has been shown to be an attractive algorithm to aid the design of microwave cavities containing highly absorbing materials.

7 ACKNOWLEDGEMENTS

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