PARAMETERIZATION AND MEASUREMENTS OF HELICAL MAGNETIC FIELDS

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Abstract

Magnetic fields with helical symmetry can be parameterized using multipole coefficients $(\tilde{a}_n, \tilde{b}_n)$. We present a parameterization that gives the familiar multipole coefficients (a_n, b_n) for straight magnets when the helical wavelength tends to infinity. To measure helical fields all methods used for straight magnets can be employed. We show how to convert the results of those measurements to obtain the desired helical multipole coefficients $(\tilde{a}_n, \tilde{b}_n)$.

1 INTRODUCTION

The magnetic field inside straight magnets can be parameterized in terms of multipole coefficients (a_n, b_n). We will present such a parameterization first. Fields of helical magnets (see Fig. 1) can be described in a similar way, by means of multipole coefficients $(\tilde{a}_n, \tilde{b}_n)$. We give a notation for the $(\tilde{a}_n, \tilde{b}_n)$ for which the (a_n, b_n) are the limiting case when the helical wave length tends to infinity. A Cylindrical coordinate system (r, θ, s) is used where s denotes the longitudinal direction.

We then assume that a magnetic field measurement device always parameterizes its measurements in terms of (a_n, b_n), and give formulae to obtain the coefficients $(\tilde{a}_n, \tilde{b}_n)$ when a helical magnetic field is measured. Three types of measurement devices are treated: rotating "radial" coils, rotating "tangential" coils and rotating Hall probes (see Fig. 2 (a), (b) and (c) respectively).

2 MAGNETIC FIELD PARAMETERIZATION

2.1 Straight Magnetic Fields

In a current free region in vacuum where the electrical field $\vec{E}$ is constant, the magnetic field $\vec{B}$ can be derived from a scalar potential ψ as $\vec{B} = -\nabla\psi$. We consider a magnet of infinite length, thus neglecting fringe fields. The symmetry condition of such an element is $\psi(r, \theta, s) = \psi(r, \theta, s+\Delta s)$ with Δs arbitrary. Therefore, the potential ψ is independent of s, $\psi(r, \theta, s) = \psi(r, \theta)$.

Having a main field $B_0 \sin\theta$ the solution of the Laplace equation $\Delta\psi = 0$ can be written as

$$\psi(r, \theta) = -B_0 \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{r^{n+1}}{r_0^n} \times$$
$$\times \left[a_n \cos\left((n+1)\theta\right) + b_n \sin\left((n+1)\theta\right) \right]. \quad (1)$$

The b_n are called "normal" and the a_n "skew" multipole coefficients. Here, the subscript "0" denotes a dipole, "1" a
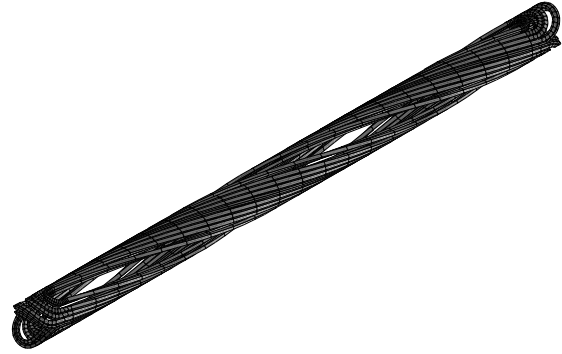
Figure 1: The conductor of a helical dipole (one helical wavelength long). This magnet will be used for proton spin manipulation in RHIC.
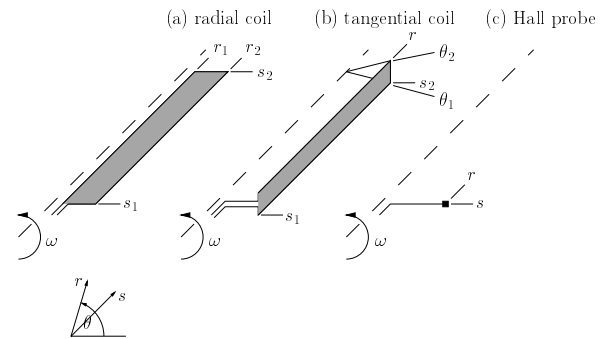

Figure 2: Three methods to measure magnetic multipole coefficients.

quadrupole etc. r_0 is a reference radius.[1]

From $\vec{B} = -\nabla\psi$ and Eq. (1) the magnetic field can be obtained. We have

$$B_r = B_0 \sum_{n=0}^{\infty} \left(\frac{r}{r_0} \right)^n \left[a_n \cos\left((n+1)\theta\right) + b_n \sin\left((n+1)\theta\right) \right], \quad (2)$$

$$B_\theta = B_0 \sum_{n=0}^{\infty} \left(\frac{r}{r_0} \right)^n \left[b_n \cos\left((n+1)\theta\right) - a_n \sin\left((n+1)\theta\right) \right], \quad (3)$$

$$B_s = 0. \quad (4)$$

2.2 Helical Magnetic Fields

We consider again a magnet of infinite length and neglect fringe fields. The symmetry condition for a helical magnet is $\psi(r, \theta, s) = \psi(r, \theta - k\Delta s, s + \Delta s)$, where Δs is arbitrary.

[1]Note that the European notation (see for example Ref. [1]) differs from the American one presented here. The transformation is $b_{n, American} = b_{n+1, European}$ and $a_{n, American} = -a_{n+1, European}$.

In other words, $\theta - ks = const.$ $k = 2\pi/\lambda$ is the wave number and λ the wave length of the helix. k shall have a positive sign for right-handed and a negative sign for left-handed helices. Introducing the new variable $\tilde{\theta} = \theta - ks$, the symmetry condition leads to a potential ψ which is only dependent on r and $\tilde{\theta}$, $\psi(r, \theta, z) = \psi(r, \tilde{\theta})$. The tilde shall remind the reader of the fact that $\tilde{\theta}$ in a helix is similar to θ in a ordinary dipole. Using $(r, \tilde{\theta})$ as coordinates and having a transverse helical main Field $B_0 \sin \tilde{\theta}$ a solution of the Laplace equation $\Delta \psi = 0$ is

$$\psi(r, \tilde{\theta}) = -B_0 \sum_{n=0}^{\infty} \frac{f_n}{k(n+1)} I_{n+1}\left((n+1)kr\right) \times$$
$$\times \left[\tilde{a}_n \cos\left((n+1)\tilde{\theta}\right) + \tilde{b}_n \sin\left((n+1)\tilde{\theta}\right) \right] \quad (5)$$

where I_n are modified Bessel functions and the f_n are defined as

$$f_n = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \frac{1}{r_0^n k^n}. \quad (6)$$

The $\tilde{b}_n$ are called "normal" and the $\tilde{a}_n$ "skew" helical multipole coefficients (with respect to the main field B_0). The subscript "0" denotes a helical dipole, the subscript "1" a helical quadrupole etc. r_0 is again a reference radius.

The factors in (5) are chosen in such a way as to obtain the potential (1) when the helical wave length tends to infinity. In this case $k \to 0$, $\tilde{\theta} \to \theta$ and the Bessel function can be approximated by $I_n(z) \approx \frac{1}{2^n} \frac{z^n}{n!}$ (cf. Ref. [2]). Now, the magnetic field can be computed as

$$B_r = B_0 \sum_{n=0}^{\infty} f_n I'_{n+1}\left((n+1)kr\right) \times$$
$$\times \left[\tilde{a}_n \cos\left((n+1)\tilde{\theta}\right) + \tilde{b}_n \sin\left((n+1)\tilde{\theta}\right) \right], \quad (7)$$

$$B_\theta = -\frac{1}{kr} B_s, \quad (8)$$

$$B_s = -B_0 \sum_{n=0}^{\infty} f_n I_{n+1}\left((n+1)kr\right) \times$$
$$\times \left[\tilde{b}_n \cos\left((n+1)\tilde{\theta}\right) - \tilde{a}_n \sin\left((n+1)\tilde{\theta}\right) \right]. \quad (9)$$

where I'_n denotes the derivative with respect to the argument of the Bessel function.

Since the Bessel function is nonlinear, a magnetic field with helical symmetry is nonlinear too, even the field of a perfect helical dipole. In addition, there is a longitudinal field component off the s-axis. Fig. 3 shows an example for the field inside a helical dipole.

3 MEASUREMENTS OF HELICAL FIELDS

While the multipole measurements with rotating coils are obtained from the field within the coil area (shaded in Fig. 2), the multipole coefficients of Hall probe measurements are obtained from the field on the circumference of a circle with given radius r. For rotating coils the magnetic flux is computed for both parameterizations (a_n, b_n)
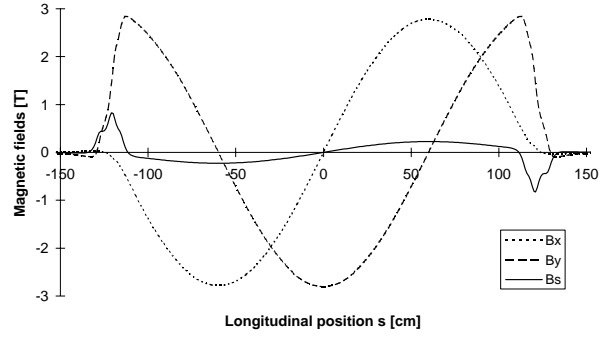

Figure 3: Magnetic field components along the s-axis with x=0 cm and y=31 cm for the magnet shown in Fig. 1. Note that end effects are not treated in this article.

and $(\tilde{a}_n, \tilde{b}_n)$. The results are stated in a form that allows a direct comparison. For Hall probe measurements the magnetic fields in both parameterizations are also stated in a form that allows a comparison. For all cases conversion formulae from (a_n, b_n) to $(\tilde{a}_n, \tilde{b}_n)$ are given in the same form.

3.1 Rotating Coils

The magnetic flux through a coil is

$$\Phi(\theta) = N \int \vec{B}(r, \theta) \cdot d\vec{a} \quad (10)$$

where N is the number of coils windings. For rotating coils one has $\theta = \omega t$ and the induced voltage $U = -d\Phi/dt$ is proportional to the angular velocity ω.

3.1.1 Radial Coils

The area of a flat rotating radial coil ranges from r_1 to r_2 and from s_1 to s_2 (cf. Fig. 2). The magnetic flux (10) through the coils is

$$\Phi(\theta) = N \int_{s_1}^{s_2} \int_{r_1}^{r_2} B_\theta(r, \theta) \, dr \, ds. \quad (11)$$

Using (3) the magnetic flux (11) for straight magnets becomes

$$\left. \begin{aligned} &\Phi(\theta) = NB_0(s_2 - s_1) \sum_{n=0}^{\infty} K_n \left[b_n \cos\left((n+1)\theta\right) - a_n \sin\left((n+1)\theta\right) \right] \\ &\text{with} \quad K_n = \frac{r_0}{n+1} \left[\left(\frac{r_2}{r_0}\right)^{(n+1)} - \left(\frac{r_1}{r_0}\right)^{(n+1)} \right]. \end{aligned} \right\} \quad (12)$$

With (8) the flux for helical fields is

$$\left. \begin{aligned} &\Phi(\theta) = NB_0(s_2 - s_1) \sum_{n=0}^{\infty} R_n \left[\hat{b}_n \cos\left((n+1)\theta\right) - \hat{a}_n \sin\left((n+1)\theta\right) \right] \\ &\text{with} \quad R_n = f_n \int_{r_1}^{r_2} \frac{1}{kr} I_{n+1}\left((n+1)kr\right) dr. \end{aligned} \right\} \quad (13)$$

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In (13) new magnetic multipole coefficients

$$\hat{a}_n = +\tilde{a}_n T_n + \tilde{b}_n S_n,$$
$$\hat{b}_n = -\tilde{a}_n S_n + \tilde{b}_n T_n. \tag{14}$$

are used for which

$$S_n(s_1, s_2) = -\frac{2}{\Delta s (n+1)k} \sin \frac{(n+1)k\Delta s}{2} \sin \frac{(n+1)k\Delta s}{2} \tag{15}$$

and

$$T_n(s_1, s_2) = +\frac{2}{\Delta s (n+1)k} \sin \frac{(n+1)k\Delta s}{2} \cos \frac{(n+1)k\Delta s}{2} \tag{16}$$

with $\Delta s = s_2 - s_1$ have been defined.

3.1.2 Tangential Coils

The magnetic flux through this type of coil is (cf. Fig. 2)

$$\Phi(\theta) = N \int_{s_1}^{s_2} \int_{\theta_1}^{\theta_2} B_r(r, \bar{\theta}) \, r d\bar{\theta} \, ds. \tag{17}$$

The difference $\Delta\theta = \theta_2 - \theta_1$ is fixed and one can assume that $\theta_1 = \theta - \Delta\theta/2$ and $\theta_2 = \theta + \Delta\theta/2$ hold. With (2) the magnetic flux in straight magnets becomes

$$\left. \begin{aligned} \Phi(\theta) &= NB_0(s_2 - s_1) \cdot \frac{2}{n+1} \sin \frac{(n+1)\Delta\theta}{2} \times \\ &\times \sum_{n=0}^{\infty} K_n \Big[a_n \cos\big((n{+}1)\theta\big) + b_n \sin\big((n{+}1)\theta\big) \Big] \\ \text{with} \quad K_n &= \frac{r^{n+1}}{r_0^n} \end{aligned} \right\} \tag{18}$$

and for helical fields one obtains with (7)

$$\left. \begin{aligned} \Phi(\theta) &= NB_0(s_2 - s_1) \cdot \frac{2}{n+1} \sin \frac{(n+1)\Delta\theta}{2} \times \\ &\times \sum_{n=0}^{\infty} R_n \Big[\hat{a}_n \cos\big((n{+}1)\theta\big) + \hat{b}_n \sin\big((n{+}1)\theta\big) \Big] \\ \text{with} \quad R_n &= f_n \, r \, I'_{n+1}\big((n+1)kr\big). \end{aligned} \right\} \tag{19}$$

The $(\hat{a}_n, \hat{b}_n)$ are defined by (14) and the (S_n, T_n) needed in this definition in (15,16).

3.2 Hall Probes

For Hall probe measurements the magnetic fields, either tangential or radial, can be compared directly.

3.2.1 Tangential Field Components

The tangential field in a straight magnet is (cf. Eq. (3))

$$\left. \begin{aligned} B_\theta(\theta) &= B_0 \sum_{n=0}^{\infty} K_n \Big[b_n \cos\big((n{+}1)\theta\big) - a_n \sin\big((n{+}1)\theta\big) \Big] \\ \text{with} \quad K_n &= \left(\frac{r}{r_0} \right)^n \end{aligned} \right\} \tag{20}$$

and in a helical field (cf. Eq. (8))

$$\left. \begin{aligned} B_\theta(\theta) &= B_0 \sum_{n=0}^{\infty} R_n \Big[\hat{b}_n \cos\big((n{+}1)\theta\big) - \hat{a}_n \sin\big((n{+}1)\theta\big) \Big] \\ \text{with} \quad R_n &= \frac{1}{kr} f_n I_{n+1}\big((n+1)kr\big). \end{aligned} \right\} \tag{21}$$

The $(\hat{a}_n, \hat{b}_n)$ are again defined by equations (14) and the (S_n, T_n) needed in this definition by equations (15,16).

3.2.2 Radial Field Components

The radial field in a straight magnet is cf. Eq. (2))

$$\left. \begin{aligned} B_r(\theta) &= B_0 \sum_{n=0}^{\infty} K_n \Big[a_n \cos\big((n{+}1)\theta\big) + b_n \sin\big((n{+}1)\theta\big) \Big] \\ \text{with} \quad K_n &= \left(\frac{r}{r_0} \right)^n \end{aligned} \right\} \tag{22}$$

and in a helical field cf. Eq. (7))

$$\left. \begin{aligned} B_r(\theta) &= B_0 \sum_{n=0}^{\infty} R_n \Big[\hat{a}_n \cos\big((n{+}1)\theta\big) + \hat{b}_n \sin\big((n{+}1)\theta\big) \Big] \\ \text{with} \quad R_n &= f_n I'_{n+1}\big((n+1)kr\big). \end{aligned} \right\} \tag{23}$$

Also in this case the $(\hat{a}_n, \hat{b}_n)$ are defined by equations (14) and the (S_n, T_n) by equations (15,16).

3.3 Conversion

It is now assumed that a device parameterizes the measured magnetic field in terms of multipole coefficients (a_n, b_n) for straight magnets. If the measured magnetic field has helical symmetry, the coefficients $(\tilde{a}_n, \tilde{b}_n)$ can be derived by comparing (12) with (13), (18) with (19), (20) with (21) or (22) with (23). One obtains for all cases

$$\begin{aligned} K_n b_n &= R_n \hat{b}_n, \\ K_n a_n &= R_n \hat{a}_n. \end{aligned} \tag{24}$$

With (14) it follows that

$$\boxed{ \begin{aligned} \tilde{a}_n &= \frac{K_n}{R_n} \cdot \frac{a_n T_n - b_n S_n}{S_n^2 + T_n^2}, \\ \tilde{b}_n &= \frac{K_n}{R_n} \cdot \frac{a_n S_n + b_n T_n}{S_n^2 + T_n^2}. \end{aligned} } \tag{25}$$

4 ACKNOWLEDGMENTS

5 REFERENCES

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[2] M. Abramowitz and I. Stegun, "Handbook of Mathematical Functions", Dover, New York (1972).