Muon Colliders - Ionization Cooling and Solenoids*

Z. Parsa, Brookhaven National Laboratory, 901A, Upton, NY 11973, USA

Abstract

For a muon collider, to obtain the needed luminosity, the phase space volume must be greatly reduced within the muon life time. The ionization cooling is the preferred method used to compress the phase space and reduce the emittance to obtain high luminosity muon beams. Alternating solenoid lattices has been proposed for muon colliders, where the emittance are large. We present an overview, discuss formalism, transfer maps for solenoid magnets and beam dynamics.

1 INTRODUCTION

Alternating solenoid lattices has been proposed as desirable for use in the earlier cooling stages of Muon Colliders, where the emittances are large. Since the minimum $\beta_\perp$'s must decrease in order to obtain smaller transverse emittances as the muon beam travels down the cooling channel. This can be done by increasing the focusing fields and/or decreasing the muon momenta, where the current carrying lithium lenses may be used (to get a stronger radial focusing and to minimize the final emittance) for the last few cooling stages. The use of 'bent solenoids' may provide the required dispersion for the momentum measurement. Where the off-momentum muons are displaced vertically by an amount:

$$\Delta y \approx \frac{P}{eB_s}\frac{\Delta P}{P}\theta_{\text{bend}}, \tag{1}$$

where B_s is the field of the bent solenoid and θ_{bend} is the bend angle. In Fig.1, the bending of the solenoid produce the dispersion required for the longitudinal to transverse emittane exchange. Where after one bend and one set of wedges the beam cross-section is asymmetric then the symmetry is restored by going through the second bend and wedge system (which is rotated by 90 degrees w.r.t. the first) [1].

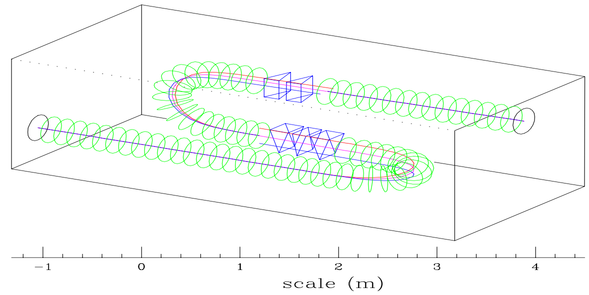

Figure 1: Example of bent solenoids and Wedges - for emittance exchange. May be used for muon collider longitudinal cooling see e.g., Refs. in [1].

2 FORMULATION AND MAPS FOR SOLENOIDS

The canonical equations in 2n-Dimensional phase space (e.g. 6 Dim., in our calculation) can be expressed as $\frac{d\psi_i}{dt} = [\psi_i, H]$, $i = 1, 2, \ldots 2n$, and in terms of the Lie transformations as

$$\frac{d\psi_i}{dt} = - : H : \psi_i, \qquad i = 1, 2, \ldots 2n \tag{2}$$

Where the Lie operator (: H :) is generated by the Hamiltonian, (H), and Lie transformation, $M = e^{-t:H:}$, could generate the solution to Eq. (2) as $\psi_i = M\psi_i(0)$, where ψ_i is the value of $\psi_i(t)$ at $t > 0$ and $\psi_i(0)$ is the initial trajectory. The interest is to find solutions to equations of motion which differ slightly from the reference orbit Thus, one can choose the canonical variables, from the values for the reference trajectory (for small deviations) and Taylor expand the Hamiltonian (H) about the design trajectory $H = H_2 + H_3 + \ldots$. Where H_n is a homogeneous polynomial of degree n in the canonical variables. After transformations to the normalized dimensionless variables, one can obtain the effective Hamiltonian H^{New}, expressed as

$$H^{\text{New}} = F_2 + F_3 + F_4 \ldots . \tag{3}$$

Thus the particle trajectory $\vec{\psi} = (X, P_X, Y, P_Y, \tau, P_\tau)$ through a beamline element of length L can be described by $\psi_i^f = - : H^{\text{New}} : \psi_i$, $i = 1, 2, \ldots 2n$. The exact symplectic map that generates the particle trajectory through that element is $M = e^{-L:H^{\text{New}}:}$, where M describes the particle behavior through the element of length L. Using the factorization and expanding H^{New} as in Eq. (3), results in

$$M = e^{-L:H^{\text{New}}:} = e^{:f_2:}e^{:f_3:}e^{:f_4:}\ldots, \tag{4}$$

(e.g., for a map through 3rd order we need to include terms of f_2, f_3, and f_4).

To illustrate the above formalism, consider the evolution of the motion of particles in an external electromagnetic field described by the Hamiltonian $H = \sqrt{m^2c^4 + c^2[(p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2]}$ $+e\phi(x, y, z; t)$, where m and q are the rest mass and charge of the particle, A and ϕ are the vector and scalar potentials such that $\vec{B} = \nabla \times \vec{A}, \vec{E} = -\nabla\phi - \nabla\vec{A}/\partial t$.

Making a canonical transformation from H to H_1 and changing the independent variable from time t to z (for convenience) for a particle in magnetic field (e.g. of solenoid) results in $p_z = [(p_z - qA_x)^2 + (p_y - qA_y)^2 + p_t^2/c^2 - m^2c^2]^{1/2}$. Where $H = -p_t$, $H_1 = -p_z$ and $t = (z/v_{0z})$ the time as a function of z. We next make a

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† E-mail: parsa@bnl.gov

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canonical transformation from H_1 to $H^{\rm New}$, with a dimensionless deviation variables (for convenience), $X = x/l$, $Y = y/l$, $\tau = c/l(t - z/v_{0z})$, $P_x = p_x/p_0$, $P_y = p_y/p_0$, $P_\tau = (p_t - p_0 t)/p_0 c$, where l is a length scale (taken as 1 m in our analysis), with $\mathbf{P} = \vec{P}_x + \vec{P}_y$ and $\mathbf{Q} = \vec{X} + \vec{Y}$ defined as two dimensional vectors [5], p_0 and $p_0 c$ are momentum and energy scales. Where p_0 is the design momentum, v_{0z} is the velocity on the design orbit and $p_0 t$ is a value of p_t on the design orbit ($p_{0t} = \sqrt{m^2c^4 + p_0^2c^2}$) (reminding that design orbit for the solenoid is along the z-axis). Thus, expanding the new Hamiltonian Eq. (3) leads to:

$$F_2 = \frac{P_\tau^2}{(2\beta^2\gamma^2)} - \frac{1}{2}B_0(\vec{Q}\times\vec{P})\cdot\hat{+}\frac{1}{8}B_0^2Q^2 + \frac{P^2}{2} \quad (5)$$

$$F_3 = \frac{P_\tau^3}{(2\beta^3\gamma^2)} - \frac{P_\tau}{2\beta}B_0(\vec{Q}\times\vec{P})\cdot\hat{z} + \frac{P_\tau}{8\beta}(B_0^2Q^2 + 4P^2) \quad (6)$$

$$\begin{aligned} F_4 = {} & \frac{P_\tau^4(5-\beta^2)}{8\beta^4\gamma^2} + \frac{P_\tau^2 Q^2 B_0^2(3-\beta^2)}{16\beta^2} \\ & - \frac{P_\tau^2}{2}(\vec{Q}\times\vec{P})\cdot\hat{z}\frac{B_0(3-\beta^2)}{2\beta^2} \\ & + \frac{P_\tau^2}{2}\,\frac{P^2(3-\beta^2)}{2\beta^2} + \frac{Q^4}{16}(B_0^4 - 4B_0B_2)/8 \\ & + \frac{Q^2}{4}\,\frac{P^2 3B_0^2}{4} + \frac{Q^2}{4}(\vec{Q}\times\vec{P})\cdot\hat{z}(B_2 - B_0^3)/4 \\ & - \frac{1}{8}(\vec{P}\cdot\vec{Q})^2B_0 - \frac{P^2}{4}(\vec{Q}\times\vec{P})\cdot\hat{z}B_0 + \frac{P^4}{8} \end{aligned} \quad (7)$$

Following the Hamiltonian flow generated by:
$H^{\rm New} = F_2 + F_3 \ldots$
from some initial ψ_0 to a final ψ_f coordinates we can calculate the transfer map M (Eq. (4)) for the solenoid. Where F_2, F_3, and F_4 would lead to the 1st, 2nd, and 3rd order maps. The effects of which can be seen from Eqs. (5–7). For example, the 2nd order effects due to solenoid transfer maps are purely chromatic aberrations Eq. (6). In addition, we note the third order geometric aberrations Eq. (7). As shown by Eqs. (5–7), the coupling between X, Y planes produced by a solenoid is rotation about the z-axis which is a consequence of rotational invariance of the Hamiltonian $H^{\rm New}$, due to axial symmetry of the solenoid field. For beam simulations, M can be calculated to any order using numerical integration techniques such as Runge-Kutta method depending on the computer memory and space available [5].

3 HIGHER ORDER KINEMATIC INVARIANTS AND CORRELATIONS:

Let $\rho(\psi)$ be the distribution of particles in phase space at any instant e.g. $d^6N = \rho(\psi)d^6\psi$, where d^6N and $d^6\psi$ are the number of particles, and small volume in the 6-dimensional phase space ($\psi = [\vec{q}, \vec{p}], q_i, p_i = 1, 2, 3$), respectively. Let $\rho(M^{-1}\psi)$ be the final distribution at the end of the system such that a set of initial moments are (j = index), defined as $k_j^0 \equiv \int \rho(\psi)F_j(\psi)d^6\psi$. Where the final moments become $k_j^f = \int \rho(\psi')F_j(M\psi')d^6\psi'$ with $F_j(M\psi) = \sum D_{j\ell}(M)F_\ell(\psi)$, ($D_{j\ell}$ is a matrix and $F_j(\psi)$ are a complete set of homogeneous polynomials.) Thus, the moment transport can be expressed in a simple form as $k_j^f = \sum_\ell F(M)_{j\ell}k_\ell^0$. $D_{j\ell}(M)$ are quadratic functions of matrix elements M_{ij}.

In 6-Dim. phase space, there are 3 functionally independent kinematic invariants made up of quadratic moments, e.g. ϵ_x^2, ϵ_y^2, ϵ_τ^2, such that $I_2(k) = \epsilon_x^2 + \epsilon_y^2 + \epsilon_\tau^2$, $I_4(k) = \epsilon_x^4 + \epsilon_y^4 + \epsilon_\tau^4$, and $I_6(k) = \epsilon_x^6 + \epsilon_y^6 + \epsilon_\tau^6$, or in general $I_n(k) = \frac{1}{2}(-1)^{n/2}t_{\rm v}(\psi J)^2$, where $\psi = 6\times6$ matrix, whose entries are moments, $\psi_{jk} = \langle\psi_j\psi_k\rangle$, with $I = 3\times3$ identity matrix and $J = -\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. E.g., $I_2(k) = \langle x^2\rangle\langle p_x^2\rangle - \langle xp_x^2\rangle + \langle y^2\rangle\langle p_y^2\rangle - \langle yp_y\rangle^2 + \langle\tau^2\rangle\langle p_\tau^2\rangle - \langle\tau p_\tau\rangle^2 + 2\langle xy\rangle\langle p_xp_y\rangle - 2\langle xp_y\rangle\langle yp_x\rangle + \ldots$. This is a generalization of 2-Dim. mean square emittance (e.g. see Refs. [2, 5]). Thus, higher order kinematic invariants (e.g. cubic and quartic moments); and correlations between various degrees of freedom may be constructed, and used as a tool, in nonlinear dynamic studies. E.G., for a beam transport system one may use an invariant: $I \equiv \langle x^2\rangle\langle p_x\rangle^2 + \langle p_x^2\rangle\langle x\rangle^2 - 2\langle xp_x\rangle\langle x\rangle\langle p_x\rangle$ constructed from a linear and quadratic moments. Noting that, the inclusion of correlations between the variables may be detrimental in (the accuracy of) beam dynamic studies.

4 MUON COOLING

Muon colliders have the potential, to provide a probe for fundamental particle physics. To obtain the needed collider luminosity, the phase-space volume must be greatly reduced within the muon life time. The Ionization cooling is the preferred method used to compress the phase space and reduce the emittance to obtain high luminosity muon beams. We note that, the ionization losses results not only in damping, but also heating: transverse heating appears due to multiple Coulomb scattering and longitudinal one is due to so named “straggling” of the ionization losses (we note that, this straggling is produced by fast “knock-on” ionization electrons), e.g. see [4]. The longitudinal muon momentum is then restored by coherent re-acceleration, leaving a net loss of transverse momentum (transverse cooling).To achieve a large cooling factor the process is repeated many times. The transverse cooling can be expressed (neglecting correlations) as

$$\frac{d\epsilon_n}{ds} = \frac{1}{\beta^2}\frac{dE_\mu}{ds}\frac{\epsilon_n}{E_\mu} + \frac{1}{\beta^3}\frac{\beta_\perp(0.014GeV)^2}{2\,E_\mu m_\mu\,L_R} + \ldots, \quad (8)$$

where β = v/c, ϵ_n is the normalized emittance, $\beta_\perp$ is the betatron function at the absorber, dE_μ/ds is the energy loss, and L_R is the radiation length of the material. The first term in this equation is the cooling term, and the second is the heating term due to multiple scattering. To minimize the heating term, a strong-focusing (small $\beta_\perp$) and a low-Z absorber (large L_R) is needed.

In obtaining Eq. 8, the correlations were neglected (as e.g. in the Status Report see [1]), e.g. $\langle xP_x\rangle = 0$), and the relation $\langle x^2\rangle = \epsilon\beta_\perp = \frac{\epsilon_n\beta_\perp}{\gamma\beta}$ was used, which can not be

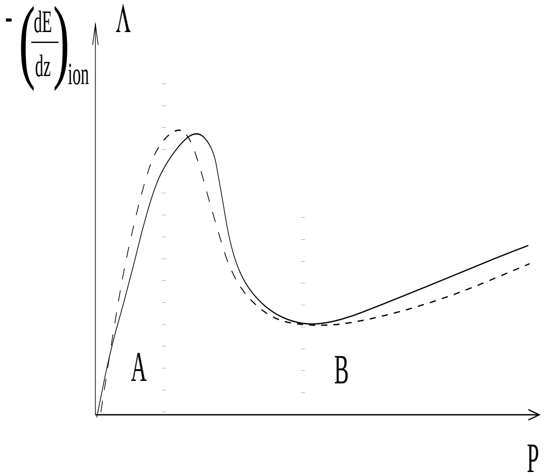

Figure 2: Schematic of the dependence of ionization losses on momenta.

assumed if the correlations are properly taken into account. Thus, if $\langle xP_x \rangle \neq 0$ then transverse cooling to be expressed as

$$\frac{d\epsilon_n}{ds} = \frac{1}{\beta^2}\frac{dE_\mu}{ds}\frac{\epsilon_n}{E_\mu} + \frac{1}{\beta^3}\frac{\langle x^2\rangle(0.014GeV)^2}{2\epsilon_n\, E_\mu m_\mu\, L_R} + ..., \quad (9)$$

As in Fig. 1, by introducing a transverse variation in the wedge (absorber) density or thickness, where there is dispersion (i.e. the transverse position is energy dependent), the energy spread, and the longitudinal emittance can be reduced. As we noted earlier, from theoretical point of view, a situation with ionization cooling completely corresponds to a situation with radiation cooling whose theory is well developed. For some standard "hierarchy" of methods for analyzing such systems see e.g. Ref. [4].

In ionization method, muons passing through a material medium lose momentum and energy through ionization interactions in transverse and longitudinal directions. The normalized emittance is reduced due to transvers energy losses. The curve in Fig. (2) shows the dependence of ionization losses on momenta. Damping rates (decrements) of individual particles in the absence of wedges (natural damping rate) are defined by the following formula:

$$\lambda_\perp = -\frac{dE}{dz}_{ion}\,\frac{1}{2\beta^2\gamma mc^2}$$

$$(10)$$

$$\lambda_\parallel = -\frac{1}{z}\,\frac{d}{dp}\left[\left(\frac{dE}{dz}\right)_{\rm ion}\,\frac{1}{v}\right]$$

Where $\lambda_\perp$ and $\lambda_\parallel$ are natural transverse and longitudinal damping respectively. Here $\left(\frac{dE}{dz}\right)_{\rm ion}$ is the ionization losses of energy, m is the muon mass, β, γ are relativistic parameters, p, v are momentum and longitudinal velocity of muons being cooled. It was established, that the sum of all increments is invariant of the cooling system: $\Lambda = 2\lambda_\perp + \lambda_\parallel$. This curve is also plotted in Fig. (2) (as the dotted line). In Fig. (2) we see that there are two natural regions for cooling: region A ("frictional cooling") and region B ("ionization cooling" for intermediate and high energies). Frictional Cooling is convenient only for cold (low energy) muons (e.g. Kinetic energy 10 to 150 KeV), and therfore it is difficult to use for high energy muon source, (in addition to big noises due to coulomb scattering etc.). Classical Ionization Cooling is useable for kinetic energy range of 30 to 100 MeV. Which due to abscence of "natural" longitudinal cooling it is necessary to use "wedges" for which R & D is needed. A propsal for such studies is being considered [1].

5 MUON COOLING "MERIT FACTOR"

Luminosity of collider L is defined by the following expression:

$$L \sim \frac{N^2 f}{g_x g_y} = \frac{N^2 f}{\epsilon_\perp^f \cdot \beta_\perp^f} \quad (11)$$

Where N = a number of muons per bunch, f = mean repetition frequency of collisions, $\epsilon_\perp^f$ = emittance at collision point and $\beta_\perp^f$ = β-function at collision point. Usually $\beta_\perp^f$ is limited by condition: $\beta_\perp^f \geq \sigma_z^f$ where σ_z^f is a longitudinal bunch size. Let us assume, that: 1) $\frac{\Delta p_f}{p}$ is known (monochromatic experiments); 2) we can redistribute emittances inside a given six-dimensional phase volume. Then, taking into account losses in the cooling system, we can rewrite Eq. (11) in the following form:

$$L \sim \frac{N_0^2 \exp\left(-\frac{2}{cT_0}\int_0^z \frac{dz}{\gamma(z)}\right) D^2}{\sqrt{V_6^N \cdot \epsilon_\parallel^f}} \cdot \left(\frac{\Delta p}{p}\right)_\parallel^f \quad (12)$$

Here "N_0" is a number of particles at an entrance of the cooling system, "exp" describes muon decay, "D" describes muon losses in cooling section, and "V_6^N" is an invariant six-dimensional phase volume of muon beam.

Thus we can introduce "merit factor" which describes a quality of muon cooling system. We obtain

$$R = \frac{D^2 \exp\left[-\frac{2}{cT_0}\int_0^z \frac{d_z}{\gamma(z)}\right]}{\sqrt{V_6^N}} \quad (13)$$

Note that, the dependence on V_6^N may be stronger. With account of all the circumstances, we can write

$$R \sim (V_6^N)^\alpha \quad (14)$$

with α in interval (0.5; 2/3). For more info. see Refs. [1-5].

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