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QUANTUM MECHANICAL ASPECTS OF THE HALO PUZZLE

Sameen A. Khan*, Modesto Pusterla† Dipartimento di Fisica Galileo Galilei, Università di Padova,
Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Padova,
Via Marzolo 8, Padova 35131, ITALY

Abstract

An interpretation of the the "halo puzzle" in accelerators based on quantum-like diffraction is given. Comparison between this approach and the others based on classical mechanics equations is exhibited.

1 INTRODUCTION AND FORMALISM

The use of a formalism that closely resembles quantum mechanics (quantum-like theory) in the description of a collective motion of dense particles, has been proposed some time ago [1]. Applications of the Schroödinger-type equations were made to many physical systems, such as plasmas, and beams in linear and circular accelerators, with a suitable definition of the the characteristic parameters.

In this note we point out that, after linearizing the Schrödinger-like equation, for beams in an accelerator one can use the whole apparatus of quantum mechanics, with a new interpretation of the basic parameters (for instance the Planck's constant $\hbar \longrightarrow \epsilon$ where ϵ is the normalized beam emittance) and introduce the propagator $K\left(x_f, t_f | x_i, t_i\right)$ of the Feynman theory for both longitudinal and transversal motion. A procedure of this sort seems particularly effective for a global description of several phenomena such as intrabeam scattering, space-charge, particle focusing, that cannot be treated easily in detail by "classical mechanics" and are considered to be the main cause of the creation of the "Halo" around the beam line with consequent losses of particles.

Let us indeed consider the Schrödinger like equation for the beam wave function

$$\mathrm{i}\epsilon\partial_t\psi = -\frac{\epsilon^2}{2m}\partial_x^2\psi + U\left(x,t\right)\psi \tag{1}$$

in the linearized case $U\left(x,t\right)$ does not depend on the density $|\psi|^2$. ϵ here is the normalized transversal beam emittance defined as follows:

$$\epsilon = m_0 c\gamma\beta\tilde{\epsilon}\,, \tag{2}$$

$\tilde{\epsilon}$ being the emittance usually considered, where as we may introduce the analog of the De Broglie wavelength as $\lambda = \epsilon/p$. We now focus our attention on the one dimensional transversal motion along the x-axis of the beam particles belonging to a single bunch and assume a Gaussian transversal profile for a particles injected in to a circular machine. We describe all the interactions mentioned above, that cannot be treated in detail, as diffraction effects by a phenomenological boundary defined by a slit, in each segment of the particle trajectory. This condition should be applied to both beam wave function and its corresponding beam propagator K. The result of such a procedure is a multiple integral that determines the actual propagator between the initial and final states in terms of the space-time intervals due to the intermediate segments.

$$\begin{aligned}
&K\left(x+x_0, T+\tau | x', 0\right)\\
&= \int_{-b}^{+b} K\left(x+x_0, \tau | x_0+y_n, T+(n-1)\tau'\right)\\
&\quad\times K\left(x+y_n, T+(n-1)\tau' |\right.\\
&\qquad\qquad \left. x_0+y_{n-1}, T+(n-2)\tau'\right)\\
&\quad\vdots\\
&\quad\times K\left(x+y_1, T | x', 0\right) dy_1 dy_2 \cdots dy_n
\end{aligned} \tag{3}$$

where $\tau = n\tau'$ is the total time of revolutions T is the time necessary to insert the bunch (practically the time between two successive bunches) and $(-b, +b)$ the space interval defining the boundary conditions. Obviously b and T are phenomenological parameters which vary from a machine to another and must also be correlated with the geometry of the vacuum tube where the particles circulate.

At this point we may consider two possible approximations for $K\left(n|n-1\right) \equiv K\left(x_0+y_n, T+(n-1)\tau' | x_0+y_{n-1}+(n-2)\tau'\right)$:

1. We substitute it with the free particle K_0 assuming that in the τ' interval ($\tau' \ll \tau$) the motion is practically a free particle motion between the boundaries $(-b, +b)$.

2. We substitute it with the harmonic oscillator $K_\omega\left(n|n-1\right)$ considering the harmonic motion of the betatronic oscillations with frequency $\omega/2\pi$

We may notice that the convolution property (3) of the Feynman propagator allows us to substitute the multiple integral (that becomes a functional integral for $n \longrightarrow \infty$ and $\tau' \longrightarrow 0$) with the single integral

$$\begin{aligned}
&K\left(x+x_0, T+\tau | x', 0\right)\\
&\quad= \int_{-b}^{+b} dy K\left(x+x_0, T+\tau | x_0+y, T\right)\\
&\qquad\times K\left(x_0+y, T | x', 0\right) dy
\end{aligned} \tag{4}$$

In this note we mainly discuss the case 1. and obtain from equation (4) after introducing the Gaussian slit $\exp\left[-\frac{y^2}{2b^2}\right]$ instead of the segment $(-b, +b)$ we obtain from

$$K\left(x+x_0, T+\tau | x', 0\right)$$

* Email: khan@pd.infn.it
† Email: pusterla@pd.infn.it

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} dy \exp\left[-\frac{y^2}{2b^2}\right] \\
&\quad \times \left\{\frac{2\pi \mathrm{i}\hbar\tau}{m}\right\}^{-\frac{1}{2}} \exp\left[\frac{\mathrm{i}m}{2\hbar\tau}(x-y)^2\right] \\
&\quad \times \left\{\frac{2\pi \mathrm{i}\hbar T}{m}\right\}^{-\frac{1}{2}} \exp\left[\frac{\mathrm{i}m}{2\hbar T}(x_0+y-x')^2\right] \\
&= \sqrt{\frac{m}{2\pi \mathrm{i}\hbar}} \left(T+\tau+T\tau\frac{\mathrm{i}\hbar}{mb^2}\right)^{-\frac{1}{2}} \\
&\quad \times \exp\left[\frac{\mathrm{i}m}{2\hbar}\left(v_0^2 T + \frac{x^2}{\tau}\right)\right. \\
&\qquad \left. + \frac{\left(m^2/2\hbar^2\tau^2\right)(x-v_0\tau)^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}}\right] \qquad (5)
\end{aligned}
$$

where $v_0 = \frac{x_0 - x'}{T}$ and $x_0$is the initial central point of the beam at injection and can be chosen as the origin ($x_0 = 0$) of the transverse motion of the reference trajectory in the test particle reference frame. Where as $\hbar$ must be interpreted as the normalized beam emittance in the quantum-like approach.

With an initial Gaussian profile (at $t = 0$), the beam wave function (normalized to 1) is

$$
f(x) = \left\{\frac{\alpha}{\pi}\right\}^{\frac{1}{4}} \exp\left[-\frac{\alpha}{2}x'^2\right] \qquad (6)
$$

r.m.s of the transverse beam and the final beam wave function is:

$$
\begin{aligned}
\phi(x) &= \int_{-\infty}^{+\infty} dx' \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{\left[-\frac{\alpha}{2}x'^2\right]} K\left(x, T+\tau; x', 0\right) \\
&= B \exp\left[Cx^2\right] \qquad (7)
\end{aligned}
$$

with

$$
\begin{aligned}
B &= \sqrt{\frac{m}{2\pi \mathrm{i}\hbar}} \left\{T+\tau+t\tau\frac{\mathrm{i}\hbar}{mb^2}\right\}^{-\frac{1}{2}} \left\{\frac{\alpha}{\pi}\right\}^{\frac{1}{4}} \\
&\quad \times \sqrt{\frac{\pi}{\left(\frac{\alpha}{2} - \frac{\mathrm{i}m}{2\hbar T} - \frac{m^2/2\hbar^2 T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}}\right)}} \\
C &= \frac{\mathrm{i}m}{2\hbar\tau} + \frac{m^2/2\hbar^2 T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}} \\
&\quad + \frac{\frac{\tau^2}{T^2}\left\{\frac{m^2/2\hbar^2 T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}}\right\}^2}{\left(\frac{\alpha}{2} - \frac{\mathrm{i}m}{2\hbar T} - \frac{m^2/2\hbar^2 T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}}\right)} \qquad (8)
\end{aligned}
$$

The final local distribution of the beam that undergoes the diffraction is therefore

$$
\rho(x) = |\phi(x)|^2 = BB^* \exp\left[-\tilde{\alpha}x^2\right] \qquad (9)
$$

where $\tilde{\alpha} = -(C+C^*)$ and the total probability per particle is given by

$$
\begin{aligned}
P &= \int_{-\infty}^{+\infty} dx \rho(x) = BB^* \sqrt{\frac{\pi}{\tilde{\alpha}}} \\
&\approx \frac{1}{\sqrt{\alpha}} \frac{mb}{\hbar T} \qquad (10)
\end{aligned}
$$

One may notice that the probability P has the same order of magnitude of the one computed in [2] if $\frac{1}{\sqrt{\alpha}}$ is of the order of b.

Similarly we may consider the harmonic oscillator case (betatronic oscillations) compute the diffraction probability of the single particle from the beam wave function and evaluate the probability of beam losses per particle.

2 PRELIMINARY ESTIMATES

Preliminary numerical estimates based on the above formulae for the two different cases of LHC [3] and HIDIF [4] designs give the following encouraging results:

LHC

Transverse Emittance, ϵ	=	3.75 mm mrad
Total Energy E	=	450 GeV
T	=	25 nano sec.
b	=	1.2 mm
P	=	3.39×10^{-5}

HIDIF

Transverse Emittance, ϵ	=	13.5 mm mrad
Kinetic Energy E	=	5 GeV
T	=	100 nano sec.
b	=	1.0 mm
P	=	2.37×10^{-3}

3 CONCLUSION

These preliminary numerical results are encouraging because they predict halo losses which seem under control. Indeed the HIDIF scenario gives a total loss of beam power per meter which is about a thousand higher than the LHC. However in both cases the estimated losses appear much smaller than the 1 Watt/m.

4 REFERENCES

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