REMARKS CONCERNING THE γ-PRODUCTION PROBABILITY OF HIGH RELATIVISTIC DIRAC-ELECTRONS IN THE POSITRON BUNCH

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Abstract: An exact relation between the γ-production probability in the Dirac-case and the corresponding probability in the Klein-Gordon-case is derived. In addition, several interesting terms are analysed when the Klein-Gordon equation is approximated up to second order in the field derivatives.

INTRODUCTION

As the main result, an exact relation between the γ-production probability occuring in beam-beam radiation in the case of Dirac electrons and the corresponding γ-production probability calculated for scalar electrons (Klein-Gordon-case) is presented. The above found relation enables the transfer of approxiamation schemes applied in the Klein-Gordon-case directly to the Dirac-case.

As an additional result, by using properties of the Airy-function, specific approximate expressions are derived in the case where derivatives of underlying variable forces up to second order are taken into account.

AN EXACT RELATION FOR THE TRANSITION PROBABILITY OCCURING IN BEAM-BEAM RADIATION

The probability for the emission of a photon produced by a high relativistic Dirac-electron traversing mainly in z-direction a positron bunch, as is well known, means considering the three cases.

Case 1: Right-handed electron emits right-handed photon and remains right-handed.

Case 2: Right-handed electron emits left-handed photon and remains right-handed.

Case 3: Right-handed electron emits right-handed photon and flips to the left-handed electron.

Thus giving:

Case 1:

$$\frac{d(W_a)}{dx}=\frac{\alpha_s}{p_{\shortparallel}^2(1-x)^3}\,\frac{1}{x}\,\frac{1}{2x}\cdot\int_{-\infty}^{+\infty}\frac{d^2\vec{k}_\perp}{(2\pi)^2}\cdot \left\|\int_{-\infty}^{+\infty}dz\hbar\tilde{k}'_+(z)e^{-\frac{i}{\hbar}\int_0^z dz's(z')}\right\|^2 \tag{1}$$

Case 2:

$$\frac{d(W_a)}{dx}=\frac{\alpha_s}{p_{\shortparallel}^2(1-x)^3}\,\frac{1}{x}\,\frac{1}{2}\cdot\int_{-\infty}^{+\infty}\frac{d^2\vec{k}_\perp}{(2\pi)^2}\cdot \left\|\int_{-\infty}^{+\infty}dz\hbar\tilde{k}'_+(z)e^{-\frac{i}{\hbar}\int_0^z dz's(z')}\right\|^2 \tag{2}$$

Case 3:

$$\frac{d(W_a)}{dx}=\frac{\alpha_s}{p_{\shortparallel}^2(1-x)^3}\,\frac{1}{x}\,2x\left(\frac{1}{x}-1\right)^2\frac{1}{4}(1-x)^2\cdot \int_{-\infty}^{+\infty}\frac{d^2\vec{k}_\perp}{(2\pi)^2}\left\|\int_{-\infty}^{+\infty}dz\frac{m}{c}e^{-\frac{i}{\hbar}\int_0^z dz's(z')}\right\|^2 \tag{3}$$

where $p_{\shortparallel}$ denotes the initial momentum (in z-direction) of the electron, $\hbar$ denotes Planck's constant, x denotes Sommerfeld's fine structure constant, x denotes the relative momentum fraction of the photon related to the initial momentum $p_{\shortparallel}$ of the electron, $\vec{k}_\perp$ denotes the component of the photon wave number vector perpendicular to the z-direction,

$$\tilde{k}_\pm(z):=\tilde{k}'_x(z)\pm i\tilde{k}'_y(z)$$

$$\hbar\tilde{\tilde{\vec{k}}}_\perp(z):=\hbar\vec{k}_\perp-(1-x)\left[\vec{p}_{i\perp}+\frac{1}{c}\int_{-\infty}^{z}dz',\vec{b}_0\right] \tag{4}$$

and where s(z) is defined by

$$s(z):=\frac{m_0^2c^2(1-x)^2+(\hbar\tilde{\tilde{\vec{k}}}_\perp(z))^2}{2x(1-x)p_{\shortparallel}}. \tag{5}$$

Apart from the above introduced abbreviations and definitions, m_0 denotes the rest mass of the electron, $\vec{b}_0$ denotes the impact parameter, $\vec{p}_{i\perp}$ denotes the perpendicular component of the initial momentum of the electron and $\vec{K}_\perp$ denotes the force the positron bunch exerts upon the electron. It is emphasized, that in the high relativistic limit, because of conservation of helicity there exists a possible fourth case:

Case 4: Right-handed electron emits left-handed photon and flips to left-handed electron which identically vanishes.

As is well known, the corresponding formula for calculating the probability fo the emission of a photon produced by a high relativistic Klein-Gordon electron traversing mainly in the z-direction a positron bunch reads:

$$\frac{d(W_a)}{dx}=\frac{d\sigma}{dxd^2\vec{b}_0}=\frac{\alpha_s}{p_{\shortparallel}^2(1-x)^{x3}}\cdot\int_{-\infty}^{+\infty}\frac{d^2\vec{k}_\perp}{(2\pi)^2}\cdot \left\|\int_{-\infty}^{+\infty}dz\hbar\tilde{\vec{k}}'_\perp(z)e^{-\frac{i}{\hbar}\int_0^z dz's(z')}\right\|^2 \tag{6}$$

Equations (1), (2), (3), (6) represent exact relations with regard to arbitrary z-dependence of the force $\vec{K}_\perp(z)$ originating from the positron bunch.

It might be interesting to find relations among (1), (2), (3), (6) or among the corresponding relevant expressions containing the integrals:

$$DR_1 := \left\| \int_{-\infty}^{+\infty} dz\, \hbar \tilde{\vec{k}}'_-(z) e^{-\frac{i}{\hbar}\int_0^z dz' s(z')} \right\|^2 \qquad (7)$$

$$DR_2 := \left\| \int_{-\infty}^{+\infty} dz\, \hbar \tilde{\vec{k}}'_+(z) e^{-\frac{i}{\hbar}\int_0^z dz' s(z')} \right\|^2 \qquad (8)$$

$$DR_3 := \left\| \int_{-\infty}^{+\infty} dz \frac{m}{c} e^{-\frac{i}{\hbar}\int_0^z dz' s(z')} \right\|^2 \qquad (9)$$

$$KG := \left\| \int_{-\infty}^{+\infty} dz\, \hbar \tilde{\vec{k}}'_\perp(z) e^{-\frac{i}{\hbar}\int_0^z dz' s(z')} \right\|^2 \qquad (10)$$

By using calculus with complex numbers as well as some vector algebra, the subsequent term can be written as:

$$\tilde{k}'_+(z_1)\tilde{k}'_-(z_2) = \tilde{\vec{k}}'_\perp(z_1)\cdot\tilde{\vec{k}}'_\perp(z_2) - - i\vec{e}_3 \cdot \left(\tilde{\vec{k}}'_\perp(z_1)\times\tilde{\vec{k}}'_\perp(z_2)\right) \qquad (11)$$

so that by using

$$DR_1 = \int\int_{-\infty}^{+\infty} dz_1 dz_2 \hbar^2 \Big(\tilde{\vec{k}}'_\perp(z_1)\cdot\tilde{\vec{k}}'_\perp(z_2) - -i\vec{e}_3 \left\langle \tilde{\vec{k}}'_\perp(z_1)\times\tilde{\vec{k}}'_\perp(z_3)\right\rangle\Big) e^{EXP} \qquad (12)$$

for the case 1, the exact relation results

$$DR_1 = KG - \hbar^2 \iint dz_1 dz_2 i\vec{e}_3 \left\langle \tilde{\vec{k}}'_\perp(z_1)\times\tilde{\vec{k}}'_\perp(z_2)\right\rangle e^{EXP} \qquad (13)$$

and as well for the remaining cases, the exact relations

$$DR_2 = KG + \hbar^2 \iint dz_1 dz_2 i\vec{e}_3 \left\langle \tilde{\vec{k}}'_\perp(z_1)\times\tilde{\vec{k}}'_\perp(z_2)\right\rangle e^{EXP} \qquad (14)$$

$$DR_3 = \frac{m^2}{c^2} \iint dz_1 dz_2 e^{EXP} \qquad (15)$$

can be presented, where the abbreviation

$$EXP = \frac{i}{\hbar}\int_{z_1}^{z_2} dz'\, s(z') \qquad (16)$$

has been introduced.

In order to arrive at the expression for the total probability, summation over final states (i.e. summation of (1), (2), (3)) has to be performed, thus leading to the central formula:

$$\frac{d(W_a)}{dx} = \frac{\alpha_s}{p_{\shortparallel}^2(1-x)^3}\frac{1}{x}\int \frac{d^2\vec{k}_\perp}{(2\pi)^2}\Big\{ KG \cdot \left\langle \frac{1}{2x} + \frac{x}{2}\right\rangle - - i\hbar^2 \int\int_{-\infty}^{+\infty} dz_1 dz_2 \vec{e}_3 \cdot \left\langle \tilde{\vec{k}}'_\perp(z_1)\times\tilde{\vec{k}}'_\perp(z_2)\right\rangle e^{EXP} \cdot \cdot \left\langle \frac{1}{2x} - \frac{x}{2}\right\rangle + \frac{m^2}{c^2}\int\int_{-\infty}^{+\infty} dz_1 dz_2 e^{EXP} \frac{(1-x)^4}{2x} \Big\}. \qquad (17)$$

Apart from the middle term

$$MT := -i\hbar \int \frac{d^2\vec{k}_\perp}{(2\pi)^2} \iint dz_1 dz_2 \vec{e}_3 \left\langle \tilde{\vec{k}}'_\perp(z_1)\times\tilde{\vec{k}}(z_2)\right\rangle e^{EXP} \qquad (18)$$

and apart from the spin-flip term

$$SPFL := \frac{m^2}{c^2}\cdot \int \frac{d^2\vec{k}_\perp}{(2\pi)^2} \iint dz_1 dz_2 e^{EXP} \qquad (19)$$

the central formula (17) permits the easy calculation of the transition probability for the Dirac-case simply by multiplying the corresponding transition probability found in the Klein-Gordon-case by the constant factor $\left\langle \frac{1}{2x} + \frac{x}{2}\right\rangle$ thus giving:

$$\frac{d(W_a)^{Dirac}}{dx} = \left\langle \frac{1}{2x} + \frac{x}{2}\right\rangle \frac{d(W_0)^{K.G.}}{dx} + \frac{\alpha_s}{p_{\shortparallel}^2(1-x)^3}\cdot \cdot \frac{1}{x}\left[\left\langle \frac{1}{2x} - \frac{x}{2}\right\rangle (MT) + \frac{(1-x)^4}{2x}(SPFL)\right] \qquad (20)$$

It should be remarked that the above mentioned relation (20) is independent of any considerations of appropriate evaluation methods of the underlying contributing integrals in the case of weakly or (more) strongly z-dependent forces $\vec{K}_\perp(z)$.

In the case of considering derivatives of the force $\vec{K}_\perp(z)$ originated from the positron bunch up to at most second order, as generally known, one arrives at:

$$\frac{d(W_a)^{K.G.}}{dx} = \left\langle \frac{1}{2x} + \frac{x}{2}\right\rangle \frac{\alpha_s m}{\hbar c^2 p_{\shortparallel}} \int_{-\infty}^{+\infty} d\bar{z} \int_u^{\infty} dv Ai(v) \cdot \cdot \left[2\cdot\frac{V}{U} - 1\right] + 2\alpha_s \frac{l_{coh}}{90\Gamma}\left(\frac{2}{U}\right) A' i(u) + + UAi(u) \left\langle \frac{1}{2x} + \frac{x}{2}\right\rangle \qquad (21)$$

The corresponding approximate expression for the spin-flip-term reads:

$$\frac{d(W_a)^{\text{Dirac-Fall}}_{\text{SPFL}}}{dx} =: \frac{\alpha_s}{p_{\shortparallel}^2(1-x)^3}\cdot\frac{1}{x}\cdot\frac{(1-x)^4}{2x}\,\text{SPFL} =$$
$$= \frac{\alpha_s m^2}{p_{\shortparallel}\hbar c^2}\int_{-\infty}^{+\infty}d\bar{z}\left[\int_U^{\infty} dv\,Ai(v)\cdot\frac{(1-x)^2}{2x}\right]+$$
$$+\int_{-\infty}^{+\infty} d\bar{z}(-1)2\alpha_s\frac{(1-x)^2}{2x}\cdot\frac{l_{coh}}{90\Gamma}\cdot \tag{22}$$
$$\cdot\left[\frac{\|\dot{\vec{K}}_\perp\|^2}{\|\vec{K}_\perp\|^2}+3\frac{\vec{K}_\perp\cdot\ddot{\vec{K}}_\perp}{\|\vec{K}_\perp\|^2}\right]\left(\frac{2}{U}A'\,i(u)+uAi(u)\right).$$

Besides the presentation of the central formula (17) as an additional result, the approximate evaluation of the integrals contained in the middle term up to the so far considered order of derivatives in the force $\vec{K}_\perp(z)$ can be treated analagously by using similar arguments leading to (21), (22) as demonstrated briefly below.

Starting from a series expansion in the variable w = z_1 - z_2 about $\bar{z}=\frac{1}{2}(z_1+z_2)$ of the subsequent expression:

$$C := -\vec{e}_3\cdot\left\langle \tilde{\tilde{\vec{k}}}'_\perp(z_1)\times\tilde{\tilde{\vec{k}}}'_\perp(z_2)\right\rangle e^{EXP} \tag{23}$$

enables one to arrive at the following relation for the middle-term:

$$MT = i\hbar^2\int_\infty\frac{d^2\tilde{\tilde{\vec{k}}}'_\perp}{(2\pi)}\int d\bar{z}dwC$$
$$TM = i\hbar^2\int_{-\infty}^{+\infty}d\bar{z}\,\frac{x^2(1-x)^2p_{\shortparallel}^2 i}{\hbar^2}\,\vec{e}_3\left\langle\vec{K}_\perp\times\dot{\vec{K}}_\perp\right\rangle\cdot \tag{24}$$
$$\frac{1}{\|\vec{K}_\perp\|^2}\cdot(-2)\cdot U\,Ai(u)$$

thus yielding the corresponding contribution of the probability:

$$\frac{d(W_a)}{dx} = \int_{-\infty}^{+\infty}d\bar{z}2\alpha_s\left\langle\frac{1}{2x}-\frac{x}{2}\right\rangle\frac{1}{3\Gamma}\vec{e}_3\cdot \tag{25}$$
$$\frac{\left\langle\vec{K}_\perp\times\dot{\vec{K}}_\perp\right\rangle}{\|\vec{K}_\perp\|^2}\frac{1}{\sqrt{u}}Ai(u)$$

or

$$\frac{d(W_a)}{dx}^{\text{Dirac-Fall-MT}} = \frac{2\alpha_s}{3}\sqrt{\frac{m^2}{p_{\shortparallel}\hbar c^2}}\cdot\left\langle\frac{1}{2x}-\frac{x}{2}\right\rangle \tag{25a}$$
$$\cdot\int_{-\infty}^{+\infty}d\bar{z}\sqrt{\frac{l_{coh}}{\Gamma}}\vec{e}_3\cdot\frac{\left\langle\vec{K}_\perp\times\dot{\vec{K}}\right\rangle}{\|\vec{K}_\perp\|^2}\frac{1}{\sqrt{u}}Ai(u)$$

where within the above presented equations the coherence length l_{coh}

$$l_{coh} := \frac{m}{\|\vec{K}_\perp(z)\|} \tag{26}$$

as well as

$$\Gamma := \frac{p_{\shortparallel}\hbar c^2}{m^2 l_{coh}} = \frac{p_{\shortparallel}hc^2\|\vec{K}_\perp\|}{m^3},$$
$$\hat{r} = \left[\frac{(1-x)\|\dot{\vec{K}}_\perp\|^2}{c^2\cdot 8\cdot\hbar\times p_{\shortparallel}}\right] \tag{27}$$

and $$m = m_0c^2,\qquad \bullet =: \frac{\partial}{\partial z} \tag{28}$$

have been introduced.

CONCLUSION

In this paper, an exact relation between the probability for the emission of a photon in the Dirac-case and the corresponding probability in the Klein-Gordon-case has been presented in the high relativistic limit.

An additional result, in the case of considering derivatives of the force $\vec{K}_\perp(z)$ up to at most second order, a detailed approximate expression of the middle term comprising the cross product $\vec{K}_\perp\times\dot{\vec{K}}_\perp$ has been obtained by using specific properties of the Airy function Ai. Obviously, this above mentioned approximate expression vanishes in the case of e.g. presupposed cylinder symmetry around the z-axis, evidently implying $\vec{K}_\perp\times\dot{\vec{K}}_\perp = 0$.