KRAKEN, a Numerical Model of RHIC Impedances

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I. THE KRAKEN MODEL

Consider two particles, with charges q_1 and q_2, circulating in the $\hat{s}$ direction, horizontally and longitudinally displaced by (x_1, z_1) and (x_2, z_2) from the center of the same bunch. The equation of motion of the second particle is

$$\frac{d^2 x_2}{dt^2} + k(s)\, x_2 = \frac{q_1 q_2}{m\gamma} W_1(s, z_1 - z_2)\, x_1 \qquad (1)$$

where $k(s)$ represents quadrupole focusing, m is the particle mass, γ is the Lorentz factor, and $\beta \approx 1$ is assumed. The "transverse wake field" W_1 characterises the way that the first particle interacts with the environment, to modify the transverse motion of the second particle. It is always positive for particles that are very close together, $W_1(s, +\epsilon) > 0$, defocusing a trailing particle that is in phase with the source particle. Causality demands that $W_1 = 0$ if $z_1 < z_2$, and if multi-turn wakes are neglected.

When a wake field generating device at $s = 0$ is short, it is natural to talk of its "transverse wake potential",

$$V_1(z_1 - z_2) \equiv \int_{device} W_1(s, z_1 - z_2)\, ds \qquad (2)$$

since then the equation of motion becomes

$$x_2'' + K(s)\, x_2 = \frac{q_1 q_2}{mc^2\gamma} V_1(z_1 - z_2)\, x_1\, \delta(s) \qquad (3)$$

where a prime denotes differentiation with respect to s. The numerical code KRAKEN models this motion in RHIC, the Relativistic Heavy Ion Collider, by giving each of N_m macroparticles an angular kick once per turn. If there are N_b ions of atomic number Z and atomic weight A in each bunch, then the net angular kick to particle i is

$$\Delta x_i' = \frac{N_b Z^2 e^2}{A m_u c^2 \gamma} \frac{1}{N_m} \sum_{j=1}^{N_m} V_1(z_j - z_i)\, x_j \qquad (4)$$

This numerical model is valid even if the device is long - for example, with a constant resistive wall wake field - if the synchrotron period in turns is long ($T_s \gg 1$), and the beta function at the kick point is the linearly weighted average beta over the device. To simplify the comparison of analytical and numerical results, it is assumed from here on that there are only two proton macroparticles ($N_m = 2$, $Z = A = 1$), so that

$$\Delta x_2' = \frac{N_b e^2}{2mc^2\gamma} V_1(z_1 - z_2)\, x_1 \qquad (5)$$

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II. GENERAL HEAD-TAIL RESULTS

If the chromaticity $\chi = dQ/d\delta$ is zero (where δ is the off-momentum parameter, $\Delta p/p$), the motion is stable for increasing N_b, until the *strong head-tail* threshold is passed. Unstable motion above this threshold has a rise time of $\tau \sim T_s$, the time scale on which the macroparticles exchange their "drive" and "response" roles. The strong head-tail instability, also known as the "transverse mode coupling", "transverse turbulent", or "transverse microwave" instability, has only been observed at electron storage rings.

The transverse motion of two macroparticles can be decomposed into "+" and "−" eigenmodes, in which the particles oscillate in or out of phase. When a small chromaticity χ is introduced, one eigenmode grows and the other is damped, with a slow timescale $\tau \gg T_s$. This is the *head-tail* instability.

The situation is conveniently parameterised by the dimensionless quantity

$$\Upsilon(\chi, \hat{z}) = \frac{\beta_D N_b Z^2 e^2}{4 A m_u c^2 \gamma} \times \qquad (6)$$
$$\int_{t=0}^{T_s/2} V_1(z_1 - z_2) \exp\left[i\, (2\,\chi\, \hat{\delta}\, T_s)\, \sin\left(\frac{2\pi t}{T_s}\right)\right] dt$$

where β_D is the Twiss function at the device [1]. The longitudinal distance between macroparticles,

$$z_1 - z_2 = 2\hat{z}\, \sin\left(\frac{2\pi t}{T_s}\right) \qquad (7)$$

is related to $\hat{\delta}$, the maximum off momentum parameter of a macroparticle, through the relationship

$$\beta_z \equiv \frac{\hat{z}}{\hat{\delta}} = \frac{\eta C T_s}{2\pi} \qquad (8)$$

where η is the slip factor and C is the circumference of the accelerator.

Two macroparticles with longitudinal amplitude $\hat{z}$ do not suffer from strong head-tail instability if

$$Re\, \Upsilon(\chi, \hat{z}) < 2 \qquad (9)$$

while the head-tail eigenmode growth rates, *per turn*, are

$$\tau_{\pm}^{-1} = \mp \frac{Im\, \Upsilon(\chi, \hat{z})}{T_s} \qquad (10)$$

These results hold for a general transverse wake field. Fortunately, τ_-^{-1} is overestimated, and in practice both modes are stabilised (above transition) by a slightly positive chromaticity.

A. STEP FUNCTION WAKE POTENTIAL

Various authors have reported analytic and simulation head-tail results for the linear (and analytically soluble) case of a step function wake potential[1], [2], [3], [4].

$$V_1(z) = V_1 \qquad z > 0 \tag{11}$$

$$V_1(z) = 0 \qquad z < 0 \tag{12}$$

In this case $\Upsilon(\chi)$ is independent of $\widehat{z}$. Fig. 1 shows the horizontal "+" mode growth rate of simulated motion as a function of chromaticity, and compares it with the prediction of Eqn. 10, when $Re\,\Upsilon(0) = 0.03$, $\widehat{\delta} = 0.001$, and $T_s = 300$ turns.

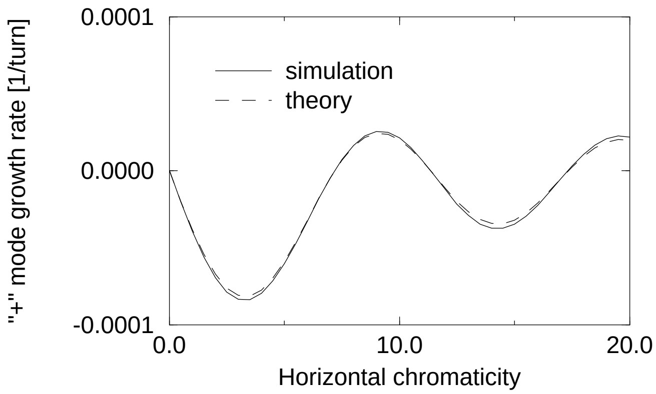

Figure. 1. The head-tail growth rate τ_+^{-1} versus chromaticity χ, for a step function wake potential.

III. NEW CRITERIA WITH MOMENTUM DEPENDENT COUPLING

It is usually implicitly assumed that linear coupling is unimportant in the head-tail effect, by treating only one transverse dimension at a time. However, the Tevatron experience is that linear coupling, and its variation with momentum, are important in routine operation close to the tune diagonal [5], [6]. When coupling *is* important, it is reasonable to conjecture that head-tail stability is only guaranteed if the eigenchromaticities are positive for all momenta within the beam [7], [8]. A weaker conjecture is that only on-momentum particles must have positive eigenchromaticities, or that it is the average eigenchromaticity over one synchrotron period that must be positive.

The extreme values for the eigenchromaticities χ_+ and χ_- (for the worst possible combination of momentum, skew quadrupole, and normal quadrupole settings) are

$$\chi_{\pm extreme} = \frac{1}{2}(\chi_x + \chi_y) \pm \frac{1}{2}\sqrt{\mathsf{k}^2 + (\chi_x - \chi_y)^2} \tag{13}$$

where the "skew chromaticity" vector k parameterises the variation of the closest approach of eigentunes, ΔQ_{min}, as a function of momentum. This is analogous to the way that the "normal chromaticities", χ_x and χ_y, parameterise the tune versus momentum, far from the tune diagonal. Insisting that both of the extreme eigenchromaticities are positive leads to the **new and strong criteria**[7], [8] that

$$\chi_x + \chi_y > 0 \tag{14}$$

$$4\chi_x\chi_y > \mathsf{k}^2 \tag{15}$$

If true, neither eigenchromaticity can ever become negative. As such, these criteria are "sufficient but often not necessary". Both χ_x and χ_y must be positive to meet the criteria, even when $\mathsf{k} = 0$, thereby recovering the standard uncoupled head-tail result (above transition).

KRAKEN has been used, with a step function wake potential, to test the new criteria. Fig. 2 shows what happens with equal design chromaticities $\chi_x = \chi_y \equiv \chi_0$, a skew chromaticity of $|\mathsf{k}| = 3.8$, and other typical Tevatron parameters [6], [8]. With the same wake potential as in Fig. 1, the simulated region of stability is shifted to the right. The horizontal "+" mode growth rate goes negative at $\chi_0 \approx 2.0$, in remarkably good agreement with the predicted condition $\chi_0 > 1.9$ that comes from the new stability criteria. Clean simulation data, as shown here, are only obtained when the initial macroparticle conditions correspond to a pure local eigenmode of the coupled system.

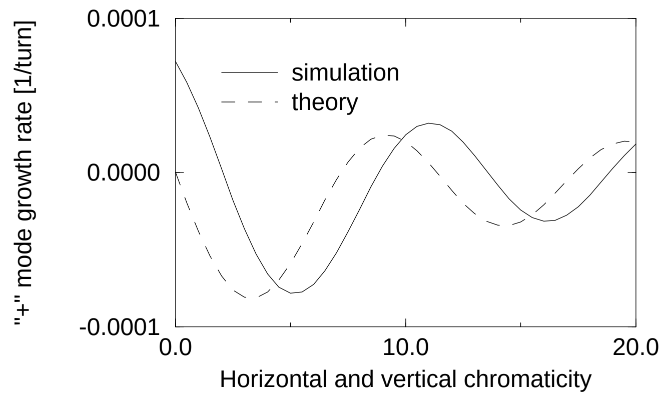

Figure. 2. The head-tail growth rate τ_+^{-1} versus common chromaticity χ_0, with momentum dependent coupling typical of Tevatron operation. The design tunes are equal.

IV. RESISTIVE WALL WAKE

The transverse resistive wall wake potential is given by

$$V_1(z) = \frac{8L}{r^4\epsilon_0\,(4\pi)^{3/2}}\; z \qquad 0 < z \ll z_c \tag{16}$$

$$V_1(z) = \frac{2L}{\pi r^3}\sqrt{\frac{c}{4\pi\epsilon_0\sigma}}\; z^{-1/2} \qquad z \gg z_c \tag{17}$$

where L, r, and σ are, respectively, the length, radius, and conductivity of the beam pipe [1]. The critical length z_c, given by

$$z_c = \left(\frac{c\epsilon_0}{\sigma}\right)^{1/3} r^{2/3} \tag{18}$$

tends to be very short. For example, $z_c = 0.12$ mm for the dominant (cold) beampipe in RHIC, with $L = 2955$ m, $r = 0.0346$ m, and $\sigma = 2.0\ \Omega^{-1}\mathrm{m}^{-1}$.

Table I lists the nominal RHIC parameters for protons at injection, when RHIC is especially vulnerable to head-tail effects. When these values are substituted into Eqn. 6, with $\widehat{\delta} = \sigma_p/p$ and $\widehat{z} = \sigma_z$, they lead to the variation of Υ with chromaticity recorded in Figure 3. Since the maximum value of $Re\Upsilon = 0.3$, RHIC is expected to be about an order of magnitude short of strong

Table I
RHIC parameters during proton injection.

Parameter	units	value
Bunch population, N_b		1.0×10^{11}
Lorentz factor, γ		31.17
Transition gamma, γ_T		22.89
Average device beta, β_D	m	30.0
Circumference, C	m	3833.8
Synchrotron period, T_s	turns	1414
RMS momentum error, σ_p/p		4.66×10^{-3}
RMS bunch length, σ_z	m	0.353

head-tail instability. Chromaticity values of $\chi \approx 2$ appear to be optimal. Figure 4 compares the growth rates predicted by Eqn. 10 with the rates measured by KRAKEN. The agreement is good in most cases, except that when $\chi \approx 1.5$, the anti-damping of the "+" mode is about 50% stronger than expected.

V. SUMMARY AND PLANS

The simulation code KRAKEN confirms analytical predictions of head-tail stability criteria, Eqns. 14 and 15, in the presence of momentum dependent linear coupling. It also confirms that resistive wall transverse wake fields are not a serious threat to strong head-tail stability in RHIC, at the vulnerable stage of proton injection.

Equation 10, derived from the perspective of two macroparticles, *potentially* offers a very convenient semi-numerical evaluation of the effects of arbitrary transverse wake potentials. It remains to be seen how well the two macroparticle results correlate with simulations using, say, 100 macroparticles.

KRAKEN is still under rapid development. Future plans are to include resonant wakefields, multiple bunches, space charge wakefields, betatron detuning, and a connection to the detailed RHIC impedance database.

References

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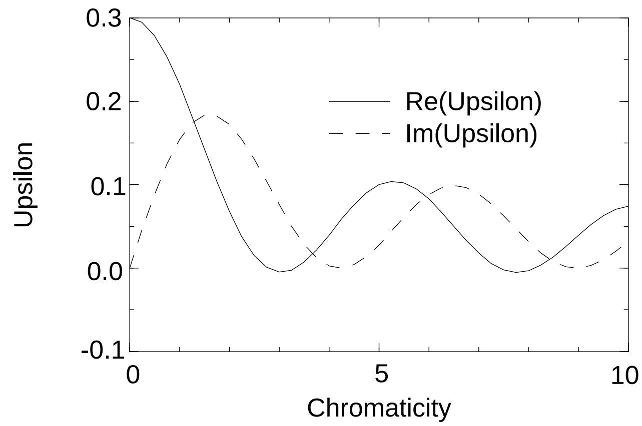

Figure. 3. Υ versus chromaticity χ for protons at injection, due to the transverse resistive wall wake.

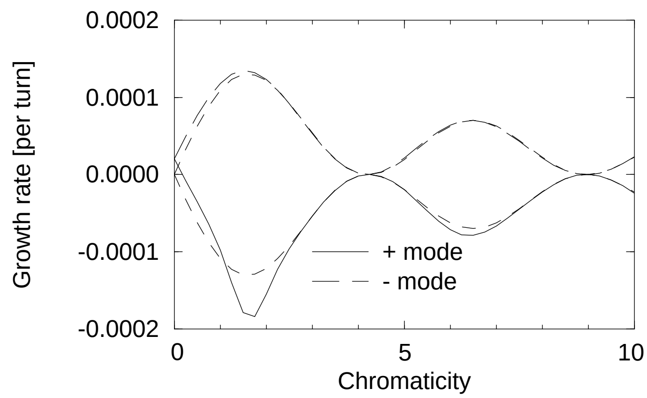

Figure. 4. Two macroparticle growth rates under proton injection conditions. Simulation observations are in good agreement with theory (short dashed lines).