REFINEMENTS TO LONGITUDINAL, SINGLE BUNCH, COHERENT INSTABILITY THEORY

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Abstract

For the case of a bunched beam confined to a quadratic potential well, we demonstrate the necessity for considering mode-coupling to correctly obtain the threshold current for the d.c. instability. Further we find the effect upon growth rate and coherent tune shift of evaluating the impedance at a complex frequency. For the case of a bunched beam confined to a cosine potential well, we give an exact analytic expression for the dispersion integral, and calculate (with no approximations), the stability diagram for the Robinson instability taking into account Landau damping. This paper comprises extracts from a lengthy internal report[1].

I. SIMPLE HARMONIC OSCILLATOR CASE

We consider the stability of a single bunch confined in a quadratic potential well that is truncated at rf-phase $x = \pm\pi$. Let ω_s be the synchrotron frequency. We shall investigate the stability of the system through use of the linearized Vlasov equation in which products of two perturbation terms will be ignored. Let the phase-space steady-state and perturbation distribution functions be $\Psi_0(J)$ and be Ψ_1, respectively. In action-angle coordinates (J,θ), the Vlasov equation becomes:

$$[\partial/\partial t \;+\; (d\theta/dt)\partial/\partial\theta]\Psi_1 \;=\; (\partial\Psi_0/\partial J)(\partial H/\partial\theta)\;. \qquad (1)$$

We shall assume Ψ_1 to have time dependence e^{st} with the complex perturbation Laplace frequency $s = \sigma + i\omega$. Let $i = \sqrt{-1}$ and take $\mathcal{R}[\ldots]$ to mean "form the real part".

Henceforward, we shall employ the symbols q and p as integer indices for Fourier harmonics.

Let $\xi = 2\pi I_{\rm d.c.}/V_{\rm rf}$.

The beam current perturbation signal is $\lambda(x,t) = \mathcal{R}[e^{st}\sum_q \lambda_q e^{iqx}]$ and leads to perturbing forces $\partial H/\partial\theta = \omega_s^2\, w(x,t)$ where the wakefields are:

$$w(x,t) = \xi\mathcal{R}[e^{st}{\textstyle\sum_q} w_q] \quad \text{with} \quad w_q = Z_q(\omega,\sigma)\lambda_q e^{iqx}\;. \qquad (2)$$

The arguments of the complex impedance Z_q are used to indicate the modulation sideband frequency. Hence $Z_{+q}(+\omega,\sigma) = Z(+q\omega_{\rm rf}+\omega,\sigma) = Z(+q\omega_{\rm rf}+\omega-i\sigma)$ is the complex impedance evaluated at the $\omega - i\sigma$ sideband of the $q^{\rm th}$ harmonic of the radiofrequency $\omega_{\rm rf}$.

As a trial solution of the Vlasov equation we take

$$\Psi_1 = \mathcal{R}[\psi\, e^{st}] \quad \text{with} \quad \psi(J,\theta) = {\textstyle\sum_m}\psi_m(J)e^{im\theta}\;; \qquad (3)$$

where m is the azimuthal mode index, and $m = 0$ is excluded. After separating the Vlasov equation, we find the radial functions

$$\psi_{+n}(J) \;=\; \frac{\omega_s^2\,\xi\, n}{[s + \, i\, n\omega_s]}\,\Psi_0' \sum_p Z_p(\omega,\sigma)\frac{1}{p}\,\lambda_p\, J_{+n}(+p\pi k)\;. \qquad (4)$$

where $k^2 = J/J_0$ and $J_0 = \omega_s\pi^2/2$ is an action value, but the $J_n(\ldots)$ with an argument are Bessel functions. The prime notation indicates a derivative with respect to action J. The Fourier harmonics are:

$$2\pi\lambda_q(n) \;=\; \textstyle\int \psi_n(J)e^{-iqx}\, e^{+in\theta} d\theta dJ\;, \qquad (5)$$

and so the eigenvalue problem is

$$\lambda_q(n) \;=\; \frac{\omega_s\,\xi\, n}{[s + \, i\, n\omega_s]}\sum_p Z_p(\omega,\sigma)\frac{1}{p}\,\lambda_p\, I_n(q,p)\;. \qquad (6)$$

Note, in the above equation the λ_p without arguments is the sum over the $\lambda_p(n)$ with arguments.

A. Single azimuthal mode and narrowband impedance

Consider the case of a solitary azimuthal $\psi(J,\theta) = \psi_m(J)e^{im\theta}$. Consider the case of a narrowband impedance such that Z_p is only significant in the vicinity of $p = q > 0$. This results in an eigenfrequency equation:

$$(s + \, i\, m\omega_s) \;=\; m\omega_s\, I_m(q,q)\,\xi\,[Z_{+q} - Z_{-q}]/q\;. \qquad (7)$$

$$\text{where} \quad I_m(q,q) = \omega_s{\textstyle\int_0^{J_0}}\Psi_0'\, J_{+m}^2(q\pi k)\, dJ\;. \qquad (8)$$

The combination ξZ and integral I_m are both dimensionless. If and only if both $Z_{-q} \neq 0$ and $Z_{+q} \neq 0$, then equation (7) has the property that if $m \Rightarrow -m$, then $s \Rightarrow s^*$.

Let the impedance $Z = R + iX$ be composed of a resistive part R and a reactive part X, then we find the eigenfrequency:

$$\omega/(m\omega_s) \;=\; I_m\,\xi[X(q\omega_{\rm rf}+\omega) \;+\; X(q\omega_{\rm rf}-\omega)]/q - 1 \quad (9)$$

$$\sigma/(m\omega_s) \;=\; I_m\,\xi[R(q\omega_{\rm rf}+\omega) \;-\; R(q\omega_{\rm rf}-\omega)]/q\;. \quad (10)$$

These equations have to be solved recursively for s . At high enough current, there is a solution with mode frequency $s = 0$, which satisfies the condition:

$$q \;=\; 2\,\xi\, X(q\omega_{\rm rf})\, I_m(q,q)\;. \qquad (11)$$

B. $\pm m$ mode-coupling and narrowband impedance

Consider the case of two azimuthal modes,

$$\psi(J,\theta) \;=\; [\psi_{+m}e^{+im\theta} + \psi_{-m}e^{-im\theta}]\;. \qquad (12)$$

Consider the case that impedance Z_p is only significant in the vicinity of $p = q > 0$. This results in an eigenfrequency equation.

$$i\frac{(s^2+m^2\omega_s^2)}{(m\omega_s)^2} = 2\xi[Z(+q\omega_{\rm rf}+\omega,\sigma)-Z(-q\omega_{\rm rf}+\omega,\sigma)]\frac{1}{q}I_m(q,q) \ . \quad (13)$$

The equation separates into imaginary and real parts as:

$$\frac{\omega^2-\sigma^2}{(m\omega_s)^2} \ = \ 1^2 - 2\,I_m\xi[X(q\omega_{\rm rf}+\omega)+X(q\omega_{\rm rf}-\omega)]/q$$

$$\omega\,\sigma/(m\omega_s)^2 \ = \ +I_m\xi[R(q\omega_{\rm rf}-\omega)-R(q\omega_{\rm rf}+\omega)]/q \ . \quad (14)$$

At high enough current, there is a solution with mode frequency $s^2 \equiv 0$.

$$q \ = \ 4\,\xi\,X(q\omega_{\rm rf})\,I_m(q,q) \ . \quad (15)$$

The value of the threshold differs by a factor 2 from the case of no mode coupling, expression (11).

II. IMPEDANCE AT COMPLEX FREQUENCY

If we continue Z into the complex plane, given the functional form $Z(\omega,\,0)$, then the response to $\exp(\sigma+i\omega)t$ is $Z(\omega,\,\sigma) = Z(\omega-i\sigma)$. Actually, one does not need to know the form, but only the derivatives of resistance R and reactance X with respect to frequency ω. We denote derivatives with respect to real angular frequency ω by ∂_ω. Let $Z(\omega,\,\sigma) = R+iX$. We may then employ the Cauchy-Riemann conditions for analytic complex functions:

$$\partial R/\partial\sigma = -\partial X/\partial\omega \quad \text{and} \quad \partial X/\partial\sigma = +\partial R/\partial\omega \ ,$$

to find the first order Taylor expansion

$$Z(\omega',\,\sigma') \ \approx \ Z(\omega',\,0) \ + \ (-i\sigma')\times\partial_\omega Z\Big|_{\substack{\sigma=0\\ \omega=\omega'}} \ . \quad (16)$$

A. Eigenvalues with narrowband impedance

Consider a narrowband impedance that is still sufficiently broad to include both the upper and lower sideband. An approximation of $[Z_{-q}-Z_{+q}]$ is

$$Z_{-q}(\omega,\sigma)-Z_{+q}(\omega,\sigma) \approx -2[iX(q\omega_{\rm rf})+(\omega-i\sigma)\partial_\omega R(q\omega_{\rm rf})] \ . \quad (17)$$

Substitution of (17) into (14) leads to the eigenvalue

$$[\omega^2+\sigma^2]/(m\omega_s)^2 \ = \ 1^2 \ - 4I_m\,\xi\,X(q\omega_{\rm rf})/q \quad (18)$$

$$\sigma/(m\omega_s) \ = \ -2\,I_m\,\xi\,m\omega_s[\partial_\omega R(q\omega_{\rm rf})]/q \ . \quad (19)$$

These forms show that, to first order, and for single bunch instability, evaluation of the impedance at a complex frequency alters the coherent tune, but does not change the growth rate.

III. SIMPLE PENDULUM OSCILLATOR

Consider the stability of a single bunch confined in a sinusoidal potential well. The unperturbed Hamiltonian is:

$$H(x,y) = y^2/2+\omega_s^2[1-\cos x] = y^2/2+2\omega_s^2\sin^2(x/2) \ . \quad (20)$$

We shall investigate the stability of a multi-particle system of oscillators through use of the Vlasov equation; the equation is simplified if we employ action-angle coordinates.

$$\sin(x/2) \ = \ k\,{\rm sn}\theta \ = \ \sqrt{J/J_0}\,{\rm sn}\theta \quad (21)$$

$$y \ = \ 2\omega_s k\,{\rm cn}\theta = 2\omega_s\sqrt{J/J_0}\,{\rm cn}\theta \ . \quad (22)$$

$J_0 = 2\omega_s$ and sn, cn dn are Jacobean elliptic functions.

The time variation of θ is $\theta = \omega_s(t-t_0)$ where t_0 is a constant of integration.

The trial solution ψ must be separable after integrating θ over the interval $[-2\mathcal{K},+2\mathcal{K}]$. Hence, we take:

$$\Psi_1 \ = \ \mathcal{R}[e^{st}\textstyle\sum_{-\infty}^{+\infty}\psi_m(J)e^{im\pi\theta/2\mathcal{K}}] \ . \quad (23)$$

After separation, we find the radial functions ψ_n :

$$\psi_n(J) = \frac{\omega_{s0}^2\,\xi\,n}{[s+in\omega_s(J)]}\left[\frac{\pi}{2\mathcal{K}}\right]\Psi_0'\sum_p Z_p(\omega)\frac{1}{p}\lambda_p\mathcal{J}_n(p,k) \ . \quad (24)$$

Using the Jacobean elliptic analogue of the Hankel transform we find a particular case of Lebedev's[2] expression:

$$\lambda_q(n) = \omega_{s0}^2\xi\sum_p Z_p(\omega)\lambda_p\left[\tfrac{n}{p}\right]\int_0^{J_0}\frac{\mathcal{J}_n(p,k)\Psi_0'\mathcal{J}_n(q,k)}{s\,+\,i\,n\,\omega_s(J)}dJ \ . \quad (25)$$

If we sum this equation over mode number n, we obtain an eigenvalue problem for the harmonics λ_q . The form factors are

$$\mathcal{J}_{+n}(+q,k)\ \times 4\mathcal{K} = \textstyle\int_{-2\mathcal{K}}^{+2\mathcal{K}} e^{-iqx}\,e^{+in\pi\theta/2\mathcal{K}}\,d\theta \quad (26)$$

$$= \int_{-2\mathcal{K}}^{+2\mathcal{K}}[{\rm dn}\theta-ik\,{\rm sn}\theta]^{+2q}\,e^{+in\pi\theta/2\mathcal{K}}\,d\theta \ ;$$

and have the properties: $\mathcal{J}_n(q,0)=0$ and $\mathcal{J}_n(q,1)=0$.

A. Narrowband impedance at cavity radio-frequency

In general, the integrals $\mathcal{J}_n(q,k)$ are awkward to evaluate analytically. To simplify, we shall consider an impedance that is significant only at the $p = \pm 1$ harmonics of the cavity radio-frequency. For odd-n we find:

$$\mathcal{J}_n(1,k) = n\left[\frac{\pi}{\mathcal{K}}\right]^2\frac{q^{n/2}}{1+q^n} \quad \text{with} \ q=\exp\left[\frac{-\pi\mathcal{K}(k')}{\mathcal{K}(k)}\right] \ . \quad (27)$$

Here q is the 'nome' and $(k')^2 = 1-k^2$. Expressions for even-n are rather complicated, but

$$\mathcal{J}_2(1,k) \approx [2\pi/\mathcal{K}]^2 q/[2(1+q)^2] \ . \quad (28)$$

B. $\pm m$ mode coupling and narrowband impedance

Previously, we saw that a mode-coupling theory is essential when the tune shifts and growth rates are comparable with the unperturbed synchrotron frequency. Consequently, we shall not bother to consider the cases of the $-m$ and the $+m$ modes in isolation. Let the mode index m be single sided and valued and take the trial distribution function (12). For the narrowband impedance we obtain the eigenfrequency equation:

$$i = \xi [Z_{+1} - Z_{-1}] 2m^2 \omega_{s0}^2 \int_0^{J_0} \frac{\omega_s(J) \Psi_0' \mathcal{J}_m^2(1,k)}{s^2 + m^2 \omega_s^2(J)} dJ \ . \quad (29)$$

Let us search for a threshold and take $s = i\omega$ pure imaginary. Let the value of action at which the integral is singular be $\tilde{J}(\omega)$ and define $\tilde{k} = \sqrt{\tilde{J}/J_0}$. Then we have the eigenequation:

$$i \ = \ \xi [Z(-\omega_{\rm rf} + \omega) \ - \ Z(+\omega_{\rm rf} + \omega)] \times [f(\omega) + ig(\omega)] \quad (30)$$

where the quantities f and g are:

$$f(\omega) \ = \ 2m^2 \omega_{s0}^2 \mathcal{P} \int_0^{J_0} \frac{\omega_s(J) \, \Psi_0' \, \mathcal{J}_m^2(1,k)}{\omega^2 \ - \ m^2 \omega_s^2(J)} \, dJ \quad (31)$$

$$g(\omega) \ = \ \omega_{s0}^2 \, \pi \Psi_0'(\tilde{J}) \, \mathcal{J}_m^2(1,\tilde{k}) \Big/ [\partial \omega_s / \partial J]_{\tilde{J}} \ . \quad (32)$$

Here $\mathcal{P}$ indicates the principal value, and g is the residue.

C. Power limited instability

For the power limited instability, the eigenfrequencies are $\omega^2 = 0$. Now zero frequency is either outside the spread of incoherent frequencies, or (for a full bucket) at the very edge of the bunch where there are no particles. Consequently, this particular instability is not Landau damped. We substitute $\omega^2 = 0$ and find the Fourier components are equal $\lambda_{-1}(-m) = \lambda_{-1}(+m)$, and that the threshold current is given by

$$1 \ = \ 4 \, \xi \, X(\omega_{\rm rf}) \, \omega_{s0}^2 \int_0^{J_0} \Psi_0' \, \frac{\mathcal{J}_m^2(1,k)}{\omega_s^2(J)} \, dJ \ . \quad (33)$$

This only differs from the linear oscillator case by virtue of the exact value of the integral.

D. Stability Diagram for $m = \pm 1$ mode coupling

We can generate constraints on the allowable impedance by considering

$$U(\omega) + iV(\omega) = \frac{i}{[f \ + \ ig]} \frac{I_{\rm a.c.}}{I_{\rm d.c.}} = \frac{2\pi I_{\rm a.c.}}{V_{\rm rf}} [Z_{-1} - Z_{+1}] \ . \quad (34)$$

Here $V_{\rm rf}$ is the cavity voltage summed about the ring, and $U, \, V$ have been normalized so that the excitation current is independent of bunch length. The method is to plot contours of constant growth rate in the $U, \, V$-plane by scanning ω. The instability threshold is given by the curve of zero growth rate and is a function of $\hat{J}$. If on the same plot, the curve $Z(\omega)$ lies wholly inside the threshold curve, then that mode is stable. As shown in figures 1–4, for the case $|m| = 1$, we have evaluated the threshold diagrams for some of the binomial functions

$$\Psi_0(J) \ \propto \ (1 - J/\hat{J})^{\alpha} \qquad \text{for} \ \ J \leq \hat{J} \quad (35)$$

The contours depict the ten cases $\hat{J}/J_0 = 0$ to 1 in steps of 0.1. The innermost and outermost contours correspond to $\hat{J} = 0$ and $\hat{J}/J_0 = 1$, respectively. For the case $\alpha < 1$ the upper intercept $\hat{V} = V(\tilde{\omega})$ must be zero, while for $\alpha > 1$ intercept $\hat{V} > 0$; for $\alpha = 1$ intercept $\hat{V}$ is undefined. For constant FW bunch length, the r.m.s. frequency spread diminishes as α increases and so the stable $U, \, V$ region grows smaller; hence the plots have different scales.

References

[1] S. Koscielniak: *Foundations of and revisions to longitudinal, single-bunch, coherent instability theory*; TRI-DN-95-13.

[2] A.N. Lebedev: *Longitudinal instability in the presence of an r.f. field* Atomnaya Energija, Vol.25, p.100 (1968).

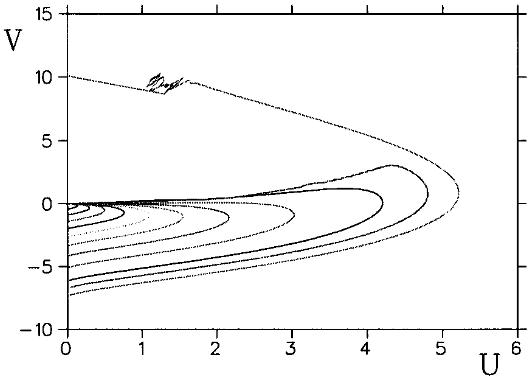

Figure 1. Stability diagram for $\alpha = 1/2$ as function of $\hat{J}$.

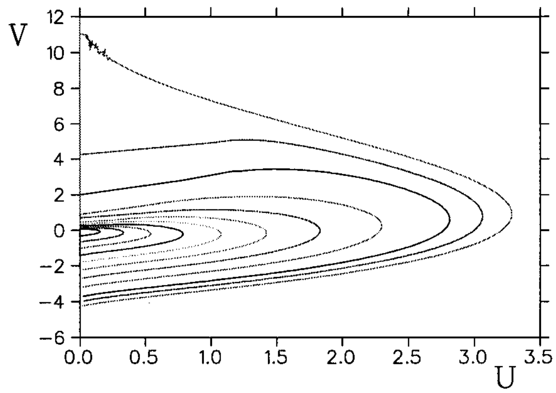

Figure 2. Stability diagram for $\alpha = 1$ as function of $\hat{J}$.

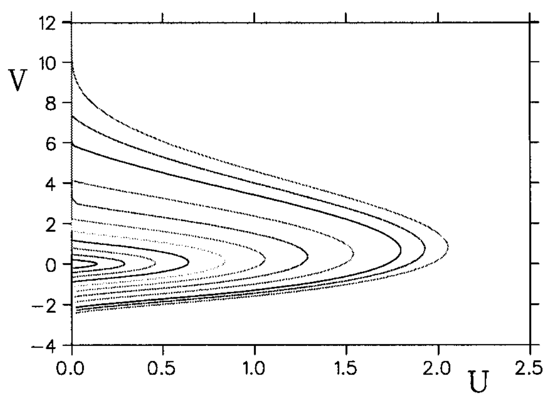

Figure 3. Stability diagram for $\alpha = 2$ as function of $\hat{J}$.

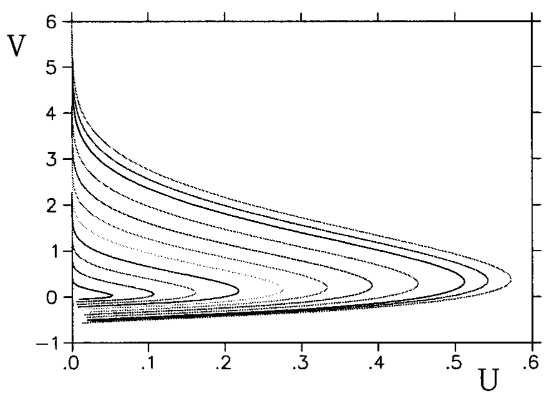

Figure 4. Stability diagram for $\alpha = 10$ as function of $\hat{J}$.