

WALL IMPEDANCES FOR LOW AND MODERATE ENERGIES

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Abstract

Properties of the wall impedance of a storage ring are studied for a low relativism, $\gamma - 1 \leq 1$. Both broad and narrow band impedances are shown to be damped at low energies, the damping factors are found. Coherent motion of a coasting beam is discussed; Landau decrements are calculated.

I. INTRODUCTION

The impedance of a vacuum chamber [1] depends on a beam velocity. This dependence vanishes in an ultrarelativistic limit, where all the fields excited by a point charge lag behind it, which is referred to as the causality principle for wake fields, see e. g. [2]. This causality principle does not work for low and moderate energies, where the relativistic factor $\gamma \simeq 1$: the Coulomb field is not here a δ -function of a longitudinal coordinate, but smoothly increase and decrease during the time $\tau \simeq r/\gamma v$, where r is an impact parameter. It follows that a broad band wall impedance $Z(\omega)$ is exponentially depressed at frequencies $\omega \geq \gamma v/b$, where b is an aperture radius. A pure space charge impedance is the only remaining at these frequencies, but it does not lead to instabilities of itself. A narrow band impedance is shown to be depressed to a lesser extent, then the broad band one, due to a rather sharp boundary of the eigenfield at the entrance, $\Delta z = b/2.4$.

A coherent motion of coasting beam is discussed; the Landau damping of both longitudinal and transverse oscillations is lost when a space charge of the beam separates coherent and incoherent frequencies [6], [9], [10]. Longitudinal and transverse Landau decrements are asymptotically calculated here for a thermal equilibrium.

II. IMPEDANCE DAMPING FACTORS

A. Wall Resistivity

Assuming the field dependence on the longitudinal coordinate and time as $e^{ik(z-vt)}$, Maxwell's equations reduce to the Poisson equation for a longitudinal electric field E_z excited by a charge linear density perturbation. Applying Leontovich boundary condition at the resistive wall surface, [3], $E_z(b) = \sqrt{-ikv/(4\pi\sigma)} B_\theta$, the electric field E_z in a perfectly conducting tube and its perturbation \tilde{E}_z due to a finite walls conductivity σ [4] can be found.

The fields E_z , \tilde{E}_z and the current perturbation J are connected by corresponding impedances, $E_z C = -Z^\parallel J$, $\tilde{E}_z C = -\tilde{Z}^\parallel J$, where C is the ring circumference, which gives:

$$\frac{Z^\parallel(kv)}{C} = \frac{2ikL}{\gamma^2 v}; \quad \frac{\tilde{Z}^\parallel(kv)}{C} = \frac{1 - ik/|k|}{2\pi b \delta \sigma} f(\kappa) \quad (1)$$

where $\delta = c/\sqrt{2\pi\sigma kv}$ is the skin depth, $L = \ln(r_{max}/a) + 1/2$ is the logarithmic factor with $r_{max} = \min(b, 1/k)$, a is

the beam radius. Here a high frequency damping factor $f(\kappa)$ has been introduced:

$$f(\kappa) = \kappa \frac{K_1(\kappa)I_0(\kappa) - K_0(\kappa)I_1(\kappa)}{I_0^2(\kappa)} = \begin{cases} 1, & \text{if } \kappa \ll 1 \\ \pi e^{-2\kappa}, & \text{if } \kappa \gg 1 \end{cases} \quad (2)$$

$K_m(\kappa)$, $I_m(\kappa)$ are modified Bessel functions.

The transverse resistive-wall impedance \tilde{Z}^\perp is found by the similar way; for any relativism $\tilde{Z}^\perp = 2\tilde{Z}^\parallel/(kb^2)$.

The real part of $\tilde{Z}^\parallel(\kappa)$ achieves its maximum at the dimensionless wavenumber $\kappa = 0.43$,

$$\text{Re}\tilde{Z}_{\max}^\parallel = 0.24Z_0(R/b)\sqrt{\gamma v/(2\pi\sigma b)},$$

$Z_0 = 4\pi/c = 377 \Omega$. Assuming $\beta = 0.4$, $\sigma = 1.3 \cdot 10^{16} \text{ s}^{-1}$, $R = 10 \text{ m}$, $b = 5 \text{ cm}$, it gives $\text{Re}\tilde{Z}_{\max}^\parallel = 3 \Omega$.

B. Broad Band Impedances

The resistive wall impedance \tilde{Z}^\parallel (Eq.1) can be represented in terms of its ultra relativistic value \tilde{Z}_{ur}^\parallel and the damping factor $f(\kappa)$ (Eq.2):

$$\tilde{Z}^\parallel(\omega) = \tilde{Z}_{ur}^\parallel f(\kappa), \quad \kappa = \omega b/(\gamma v). \quad (3)$$

The factor $f(\kappa)$ reflects the strong decrease of the incident field energy flux at the distance $r = b$ as $\sim \exp(-2kb/\gamma)$. The incident field of the beam, damped near the walls as $f^{1/2} \sim \exp(-kb/\gamma)$, produces proportionally to itself a perturbation of the surface current density. The energy loss which is a product of the field \tilde{E}_z and the current induced, is damped quadratically, as $f^{1/2} \cdot f^{1/2} = f \simeq \pi \exp(-2kb/\gamma)$. The energy loss is proportional to the real part of the impedance, so the last one is damped in the same way.

Thus, Eq.(3) follows from the simple physical consideration, which is the same for all the types of the wall imperfections. The only importance is that field perturbations introduced by these imperfections are small, which practically can be applied to all broad band wall impedances.

The real part of the broad band impedance has non-zero value only above the low frequency cut-off: $\omega \geq \omega_c$. For the circular cross-section of the vacuum chamber $\omega_c = 2.4c/b$. Taking into account that $\omega = kv$, the damping factor f near the lowest possible frequency (cut-off) occurs to be:

$$f \simeq \pi \exp(-4.8/(\gamma\beta))$$

So, the broad band impedance of wall imperfections (discontinuities, shallow cavities, irises, etc.) cannot play any role for insufficiently relativistic particles. Even for $\beta = 0.4$, the damping factor $f = 2 \cdot 10^{-5}$, which makes the broad band impedance completely negligible.

For a *transverse broad band impedance*, \tilde{Z}^\perp the low energy suppression is actual even to the greater extent. The cut-off frequency for unsymmetrical modes of the vacuum pipe ω_c^\perp is significantly higher, than for symmetrical ones, $\omega_c^\perp = 3.8c/b$ for

the circular pipe, which gives the following transverse broad band damping factor:

$$\tilde{Z}^\perp(\omega) = \tilde{Z}_{ur}^\perp(\omega) f_\perp(\kappa), \quad f_\perp(\kappa) \simeq \exp(-7.6/(\gamma\beta)). \quad (4)$$

C. Narrow Band Impedances

Another possibility for energy loss and for instabilities is connected with the radiation in low frequency ($\omega < \omega_c$) modes of some elements of the vacuum chamber. At the resonance, the impedance is pure real, it achieves here its maximum, the shunt impedance $R_s(v)$. The velocity dependence of the shunt impedance is determined by a transite time factor $T(v)$ (see, e. g. [5]):

$$R_s(v) = R_s(c) \left| \frac{T(v)}{T(c)} \right|^2, \quad T(v) = \int_{-\infty}^{\infty} E(z) e^{-i\omega z/v} dz, \quad (5)$$

where $E(z)$ is an arbitrary normalized eigenfield distribution. It follows the estimation:

$$\frac{R_s(v)}{R_s(c)} = [\beta \sin(\omega g/(2v)) / \sin(\omega g/(2c))]^2 \left\{ \frac{\delta\psi}{\sinh(\delta\psi)} \right\}^2,$$

where $\psi = \omega g/v$, g is a gap length, $\delta\psi = \pi\omega\Delta z/v$, longitudinal variation of the eigenfield in the gap assumed to be $\ll \omega/v$. The $\{\dots\}$ -factor reflects an influence of a width of the field boundary $\Delta z = b/2.4$ at the entrance and exit of the narrow band element.

III. COHERENT STABILITY

A. Longitudinal Oscillations

The dispersion relation can be found from the kinetic equation, which is the Vlasov equation plus cooling-diffusion Fokker-Planck term [6]. In the reference frame:

$$\frac{\partial f}{\partial t} + w \frac{\partial f}{\partial z} + \frac{Z_i e}{M_{\parallel}} E \frac{\partial f_0}{\partial w} = \frac{\partial}{\partial w} \left(\lambda_{\parallel} w f + d_{\parallel} \frac{\partial f}{\partial w} \right), \quad (6)$$

where f_0 is the beam phase density, f is its perturbation, w is a deviation of the particle velocity from the beam velocity v , $M_{\parallel} = M_i(1/\gamma^2 - 1/\gamma_t^2)^{-1}$ is a longitudinal mass of an ion of the beam, $M_i = A_i M_p$ is its mass, A_i and Z_i are the mass and charge numbers, λ_{\parallel} is the cooling rate and d_{\parallel} is the diffusion coefficient. Assuming the oscillations to be mainly determined by the space charge impedance Z^{\parallel} , the corrections introduced by the cooling, temperature (Landau damping) and the impedance \tilde{Z}^{\parallel} can be found as perturbations. It gives the following dispersion relation for longitudinal coherent modes in a coasting beam [7]:

$$\Omega(k) = \pm k u_{\parallel} \left\{ 1 + \frac{\pi i}{2} \frac{k}{|k|} u_{\parallel}^2 f_0'(\Omega/k) - i \frac{\tilde{Z}^{\parallel}}{2|Z^{\parallel}|} \right\} - i \frac{\lambda_{\parallel}}{2},$$

$$u_{\parallel} = c \sqrt{2\bar{\rho} r_0 L (\gamma^{-2} - \gamma_t^{-2})}, \quad r_0 = Z_i^2 e^2 / (M_i c^2), \quad (7)$$

where $\bar{\rho} = N/C$ is a linear density of the beam.

According to Eq.(7), instabilities caused by the real part of the impedance $\text{Re}\tilde{Z}^{\parallel}$ can be avoided due to the Landau damping or due to cooling. In the first case longitudinal temperature of the beam must be sufficiently high. Assuming the distribution function to be Gaussian, $f_0(u) = (2\pi\Delta w^2)^{-1/2} \exp(-u^2/2\Delta w^2)$, the stability condition can be expressed as:

$$\frac{u_{\parallel}^2}{\Delta w^2} \leq 2 \ln \left(\sqrt{\frac{\pi}{2}} \frac{u_{\parallel}^3}{\Delta w^3} \frac{|Z^{\parallel}|}{\text{Re}\tilde{Z}^{\parallel}} \right) \quad (8)$$

The factor $|Z^{\parallel}|/\text{Re}\tilde{Z}^{\parallel}$ is usually pretty large. Therefore the stability condition (8) is almost independent on the impedance. In terms of a temperature it can be expressed as:

$$T^{\parallel} \geq T_{th}^{\parallel}(\mathbf{K}) = 0.02 \frac{I(\mu\text{A}) Z_i L}{\beta(1 - \gamma^2/\gamma_t^2)} f_Z \quad (9)$$

where the factor $f_Z \approx 1$ reflects the weak logarithmic dependence on the impedance. For instance, longitudinal oscillations in a $1 \mu\text{A}$ beam of Li_7^{+1} [8] with $\beta = 0.06$, $L = 4$, will be Landau-damped if its longitudinal temperature $T^{\parallel} \geq T_{th}^{\parallel} = 1.6 \text{ K}$. For 1 mA beam of C_{12}^{+6} at TSR with $\beta = 0.041$ the threshold temperature calculated from (9) with $L = 5$, $f_Z = 1.5$ is: $T_{th}^{\parallel} = 2 \cdot 10^4 \text{ K}$, which is rather close to the experimental value of $T^{\parallel} = 3 \cdot 10^4 \text{ K}$.

This limit can be removed if the cooling rate is high enough:

$$\lambda_{\parallel} > 2\Lambda_k = \frac{u_{\parallel} \beta \text{Re}\tilde{Z}_k^{\parallel}}{L R Z_0} \quad (10)$$

If this is satisfied for all the wavenumbers $k = n/R$, the oscillations are stable even without Landau damping. For a given λ_{\parallel} the restriction (10) can be treated as a safe condition imposed on the impedance \tilde{Z}^{\parallel} . Assuming $\lambda_{\parallel} = 10 \text{ s}^{-1}$, for the Li_7^{+1} beam it gives: $\text{Re}\tilde{Z}^{\parallel} < 10 \text{ K}\Omega$.

B. Transverse Oscillations

Transverse oscillations of ions of a coasting beam can be described by the following equation:

$$\ddot{x} + \omega_b^2 x - \omega_{sc}^2 (x - \bar{x}) = (\bar{\rho} r_0 c^2 / \gamma) \int W_{\perp}(s-s') \bar{x}(s') ds' - \lambda_{\perp} \dot{x}, \quad (11)$$

where x, \bar{x} are transverse coordinates of an individual ion and a center of mass of the beam, $\dot{x} = dx/dt = (-i\Omega + in\omega_0\eta w/v)x$ is a transverse velocity of the ion, $\eta = \gamma^{-2} - \gamma_t^{-2}$, w is a deviation of a longitudinal velocity from its average value v , $\omega_b = \omega_{b0} + \xi\omega_0 w/v$ is a betatron frequency with a chromaticity accounted, $\bar{\rho}$ is a linear density of the beam, ω_{sc}^2 describes a space charge defocusing, $W_{\perp}(s)$ is a transverse wake function, λ_{\perp} is a transverse cooling rate. From here, a dispersion equation follows:

$$1 = \int \frac{dI dw f_{\perp 0}(I) f_{\parallel 0}(w) (\omega_{sc}^2(I) + \omega_c^2)}{(\Omega - n\eta\omega_0 \frac{w}{v} + i \frac{\lambda_{\perp}}{2})^2 - (\omega_{b0} + \omega_0 \xi \frac{w}{v})^2 + \omega_{sc}^2(I)}, \quad (12)$$

where $\omega_c^2 = 2i\bar{\rho}r_0c\omega_0\gamma^{-1}(Z^\perp/Z_0)$, I is a transverse action, $f_{\perp 0}(I), f_{\parallel 0}(w)$ are unperturbed distribution functions, $\int dI dw f_{\perp 0}(I) f_{\parallel 0}(w) = 1$. For small Landau damping and cooling corrections, the equation (12) can be solved:

$$\begin{aligned}\Omega &= -(\omega_{b0} + \Delta\Omega_c)n/|n| - i\lambda_\perp/2 - i\delta_\perp \\ \Delta\Omega_c &= i\frac{\bar{\rho}r_0c}{\gamma Q_b} \frac{Z^\perp}{Z_0}, \quad Q_b = \frac{\omega_{b0}}{\omega_0} \\ \delta_\perp &= \pi\Delta\omega_L \int dI dw f_{\perp 0}(I) f_{\parallel 0}(w) \Delta\omega_{sc}(I) \cdot \\ &\quad \cdot \delta(\Delta\omega_{sc}(I) + \text{Re}\Delta\Omega_c - \xi\omega_0 w/v + |n|\eta\omega_0 w/v).\end{aligned}\quad (13)$$

Here $\Delta\omega_{sc}(I) = \omega_{sc}^2(I)/(2\omega_{b0})$ is an incoherent space charge betatron shift for ions with the action I , $\Delta\omega_L = \Delta Q_{L\omega_0} = \Delta\omega_{sc}(0)$ is the Laslett shift.

The Landau damping term δ_\perp is seen to be lost when the incoherent and coherent frequencies are separated more than a width of the distribution. Typically, at low relativism $\Delta\omega_L \gg \Delta\Omega_c$, which follows the stability threshold for the Laslett shift [9], [10]. Actually, the situation is similar to the longitudinal case, where the space charge separates velocities of particles and waves, also causing the threshold to be almost independent on the ring impedance (Eq.9).

The Landau decrement δ_\perp can be asymptotically calculated from Eq.(13) for a thermal equilibrium, with some longitudinal and transverse temperatures $T_{\parallel} = M\Delta w^2$, T_{\perp} :

$$\delta_\perp = \pi\sqrt{\frac{2}{3}}\Delta\omega_L p^{2/3} \exp(-3p^{2/3}), \quad p = \frac{\Delta Q_L}{|-\xi + |n|\eta|} \frac{v}{\Delta w}.\quad (14)$$

From here, the stability condition for the Landau suppression of the instability can be presented as:

$$\Delta Q_L \leq \Delta Q_{th} \approx 0.2|-\xi + |n|\eta| \frac{\Delta w}{v} \ln^{3/2} \left(\frac{5\Delta\omega_L}{\Lambda_n} \right), \quad (15)$$

where $\Lambda_n = \text{Im} \Delta\Omega_c$. The limit value for the Laslett tune shift is determined by the mode with the minimum longitudinal number n , which has the increment higher than cooling decrement: $\Lambda_n > \lambda_\perp/2$. Due to a large argument of the logarithm ($\sim 10^5 \div 10^7$), the result is almost insensitive on it; the logarithmic factor itself is a large constant: $\ln^{3/2}(\dots) \approx 50 \div 60$. Assuming the cooling rate to be smaller than long wave ($n \leq |\xi|$) increments, the threshold Laslett shift is found:

$$\Delta Q_{th} \approx 10|\xi|\Delta w/v. \quad (16)$$

Taking as an example typical for TSR at the threshold $\Delta w/v = 10^{-3}$ and $|\xi| = 5$, it gives $\Delta Q_{th} = 0.05$, which is close to the observed value [11]. The restriction on the Laslett tune shift (15) is removed for sufficiently low currents, with the cooling rate higher than coherent increments: $\lambda_\perp > 2\Lambda$. For the mentioned Li beam, with $\lambda_\perp = 1s^{-1}$, it gives the restriction for the impedance: $\text{Re}Z^\perp < 15M\Omega/m$.

IV. CONCLUSIONS

The concept of the storage ring impedance, developed mainly for ultrarelativistic beams cannot be at once applied to low and moderate energy cases. The reason is that the causality principle for wake fields generally is not valid here, which cause an exponential damping of impedances above a certain threshold. The approach suggested gives the possibility to find the wall impedance for an arbitrary energy if its ultrarelativistic value is known. At low energies, $\beta \leq 0.5 \div 0.8$ the broad-band wall impedance is exponentially damped and too weak to cause instabilities of a cooled beam. A narrow-band impedance could be dangerous if its eigenfrequency ω_r is rather low: $\omega_r b/v \leq 1$.

Space charge of a beam switches-off the Landau damping, above certain thresholds, for both longitudinal and transverse oscillations. For typical cooling rates, low current (μA) beams could be cooled below the thresholds.

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