WAKE FIELD AND THE DIFFRACTION MODEL DUE TO A FLAT BEAM MOVING PAST A CONDUCTING WEDGE

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I. INTRODUCTION

To clean a beam of its excessive tail particles, one often uses a collimator. If the beam intensity is high enough or if the beam is brought too close to the collimator, however, the wake fields generated by the beam-collimator interaction can cause additional beam tails to grow, thus defeating, or even worsening the beam-tail cleaning process.

The wake field generated by a sheet beam moving past a conducting wedge has been obtained in closed form by Henke using the method of conformal mapping.[1] This result is applied in the present work to obtain the wake force and the transverse kick received by a test charge moving with the beam. For the beam to be approximated as sheet beams, it is assumed to be flat and the collimator is assumed to have an infinite extent in the flat dimention. We will derive an exact expression for the transverse wake force delivered to particles in the beam bunch. Implication of emittance growth as a beam passes closely by a collimator is discussed.

We consider two idealized wedge geometries. Section 2 is when the wedge has the geometry as a disrupted beam pipe. Section 3 is when it is like a semi-infinite screen. Unfortunately we have not solutions for more realistic collimator geometries such as when it is tapered to minimize the wake field effects. Our results however should still serve as pessimistic limiting cases.

An interesting opportunity is offered by our exact calculation of the wake fields: it can be used to confront the diffraction model[2,3,4] used to estimate the high frequency impedance of a cavity structure. It is shown that the field pattern, as well as the impedance, agree with those obtained by the diffraction model in appropriate limits.

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II. DISRUPTED PIPE

Consider a metal wedge and a rod beam as shown in Fig.1(a). Both the wedge and the beam are considerd to be infinitely long in the z-direction. The beam has a line charge density λ_0 and is assumed to move with the speed of light in the x-direction. Following [1], we define

$$\lambda = \frac{\pi}{2\pi - \theta} \quad \text{and} \quad R = \left(\frac{ct - \sqrt{c^2t^2 - r^2}}{r} \right)^{\lambda} \tag{1}$$

The parameters have the ranges $0 < \phi < 2\pi - \theta$, $1 > R > 0$, $\pi > \theta > 0$, and $1 > \lambda > \frac{1}{2}$. We have shown the coordinates in Fig.1(a).

In the region $r < ct$ (inside the "light cylinder"), the electromagnetic field components are found by an extension of the analysis of [1] to be

$$\begin{aligned} E_r &= -8\lambda\lambda_0 \sin\pi\lambda \sin\lambda\phi \frac{\frac{1}{R} - R}{rQ} \\ E_\phi &= 8\lambda\lambda_0 \sin\pi\lambda \frac{\frac{ct}{r}[2\cos\pi\lambda - (\frac{1}{R} + R)\cos\lambda\phi]}{Q\sqrt{c^2t^2 - r^2}} \\ B_z &= \frac{r}{ct} E_\phi, \qquad E_x = -E_r\cos\phi + E_\phi \sin\phi \\ E_y &= -E_r \sin\phi - E_\phi \cos\phi \end{aligned} \tag{2}$$

where $Q = (\frac{1}{R} - R)^2 \sin^2\pi\lambda + [(\frac{1}{R} + R)\cos\pi\lambda - 2\cos\lambda\phi]^2$. The fields are independent of the y-separation between the rod beam and the wedge.

Consider a test charge e which follows behind the rod beam at a distance D ($D > 0$) and has a vertical distance Y from the edge of the wedge ($Y > 0$), as shown in Fig.1(b). Let the test charge move with the beam at the speed of light. The Lorentz force seen by the test charge has the components

$$F_x = eE_x, \quad F_y = eE_y - eB_z, \quad \text{and} \quad F_z = 0 \tag{3}$$

We want to calculate the integrated longitudinal and transverse impulses received by the test charge as it passes by the wedge.

When $ct \to \infty$, the test charge sees $E_x \to 1/\sqrt{ct}$. It follows that the longitudinal impulse received by the test charge is infinite. This means the beam loses an infinite amount of energy to generate the wake field. The infinity does not go away with a finite wedge angle θ, or with a finite bunch length in x; it comes from the infinite bunch width in z.

The total transverse impulse, on the other hand, converges and gives the surprisingly simple result

$$c\Delta p_y(Y, D) = \int_{(Y^2+D^2)/2D}^{\infty} F_y d(ct) = 2\pi e\lambda_0 \tag{4}$$

The transverse impulse is independent of Y or D. It is even independent of the wedge angle θ.

If the beam has a surface charge density $\Sigma(x)$, its wake effects can be obtained from the rod beam result by superposition. Consider a beam particle at location x relative to the beam center. It receives a transverse impulse from all particles in front of it. Thus

$$c\Delta p_y(x) = 2\pi e \int_x^{\infty} dx' \Sigma(x') \tag{5}$$

The previous results become simpler for the case of infinitely thin wedge when $\theta = 0$ (or $\lambda = \frac{1}{2}$):

$$B_z = -\frac{4\lambda_0 \cos\frac{\phi}{2}}{r\sqrt{2(\frac{ct}{r} - 1)}(\frac{ct}{r} + \cos\phi)}$$

$$E_x = -\frac{4\lambda_0 \sin\frac{\phi}{2}}{r\sqrt{2(\frac{ct}{r}-1)}}, \quad E_y = \frac{4\lambda_0 \cos\frac{\phi}{2}(\frac{ct}{r}-1+\cos\phi)}{r\sqrt{2(\frac{ct}{r}-1)}(\frac{ct}{r}+\cos\phi)}$$

$$E_r = \frac{ct}{r}B_z, \quad E_\phi = -\frac{4\lambda_0}{r\sqrt{2(\frac{ct}{r}-1)}}\sin\frac{\phi}{2}\frac{\frac{ct}{r}-1}{\frac{ct}{r}+\cos\phi}$$

$$F_x = -\frac{4e\lambda_0 \sin\frac{\phi}{2}}{r\sqrt{2(\frac{ct}{r}-1)}}, \quad F_y = \frac{4e\lambda_0 \cos\frac{\phi}{2}}{r\sqrt{2(\frac{ct}{r}-1)}} \tag{6}$$

The sign of the Lorentz force is such that the test charge always sees a retarding force ($F_x < 0$). Also the transverse deflecting force deflects it toward the plate ($F_y > 0$).

III. SEMI-INFINITE SCREEN

The arrangement of the wedge and a rod beam is now shown in Fig.2. We have $\theta < \frac{\pi}{2}$ and $\frac{1}{2} \le \lambda \le \frac{2}{3}$. For a rod beam, inside the light cylinder, the field components are found by an extention of [1] to be

$$\begin{aligned} E_r &= -4\lambda_0\lambda \sin\frac{\lambda\pi}{2}\frac{1}{r}[2f(\lambda\phi) \\ &\quad + f(\lambda\phi-\lambda\pi) + f(\lambda\phi+\lambda\pi)] \\ E_\phi &= -4\lambda_0\lambda \sin\frac{\lambda\pi}{2}\frac{ct}{r\sqrt{c^2t^2-r^2}}[2g(\lambda\phi) \\ &\quad + g(\lambda\phi-\lambda\pi) + g(\lambda\phi+\lambda\pi)] \\ B_z &= \frac{r}{ct}E_\phi, \qquad E_x = -E_r\sin\phi - E_\phi\cos\phi \\ E_y &= E_r\cos\phi - E_\phi\sin\phi \end{aligned} \tag{7}$$

where

$$\begin{aligned} f(u) &= \frac{(\frac{1}{R}-R)\sin u}{(\frac{1}{R}-R)^2\sin^2\frac{\pi\lambda}{2} + [(\frac{1}{R}+R)\cos\frac{\lambda\pi}{2} - 2\cos u]^2} \\ g(u) &= \frac{(\frac{1}{R}+R)\cos u - 2\cos\frac{\lambda\pi}{2}}{(\frac{1}{R}-R)^2\sin^2\frac{\lambda\pi}{2} + [(\frac{1}{R}+R)\cos\frac{\lambda\pi}{2} - 2\cos u]^2} \end{aligned} \tag{8}$$

The transverse impulse as seen by a test charge shown in Fig.2(b) is found to be

$$c\Delta p_y(Y,D) = \int_{(Y^2+D^2)/2D}^{\infty} F_y d(ct) = \pi e\lambda_0 \tag{9}$$

Again this is independent of Y, D, and θ. Note Eq.(9) is exactly half of Eq.(4). It also follows that for a beam with surface charge density $\Sigma(x)$, a particle at position x receives a transverse kick which is half od Eq.(5).

IV. GENERAL WAKE CONSIDERATIONS

The fact that the integrated transverse wake force is independent of the transverse and the longitudinal locations of the test charge has its origin in the Maxwell equations. By our assumptions, we know that (a) the beam current density $\vec{j}$ and the charge density ρ are related by $\vec{j} = c\rho\hat{x}$, (b) the only non-vanishing field and force components are B_z, E_x, E_y, F_x and F_y, (c) all quantities do not depend on z, and (d) the integrated field and force components $\vec{\mathcal{E}}, \vec{\mathcal{B}}, \vec{\mathcal{F}} \equiv \int(\vec{E},\vec{B},\vec{F})d(ct)$ depend on x and t only through $x-ct$. By linearly combining the Maxwell equations into equations in terms of $e\mathcal{B}_x$, $\mathcal{F}_x$, $\mathcal{F}_y$, and $\mathcal{F}_z$, we find

$$\frac{\partial}{y}\mathcal{F}_x = \frac{\partial}{x}\mathcal{F}_y = \frac{\partial}{y}\mathcal{F}_y = 0 \tag{10}$$

This means $\mathcal{F}_y$ can not depend on x or y, i.e. it has to be constant. Also, $\mathcal{F}_x$ does not depend on y, although it can depend on x. This conclusion is valid independent of the boundary conditions, as long as the boundary is independent of the z-coordinate.

It can also be shown from a general wake consideration [5] that the wake function does not depend on Y. Observing that the wake integral scales with the ratio of Y and D, it can be concluded that the wake integral must also not depend on D. The specific value of the wake integral then follows easily by setting $Y = 0$ and $\phi = \pi$.

V. THE DIFFRACTION MODEL

A diffraction model has been proposed and used to estimate the high frequency impedance of a cavity structure in the beam pipe.[2,3,4] Consider a cylindrical beam pipe of radius b and a cavity structure of total gap length g, and a beam current $\sim e^{ik(x-ct)}$. This model suggests: (a) The wake field created as the beam passes the entrance edge of the cavity populates mainly the region in the forward direction into the open cavity space. By the time the wake field reaches the exit edge of the cavity, the radial spread of the region is

$$\Delta y \sim \sqrt{g/k} \tag{11}$$

(b) The longitudinal impedance at high frequencies is given by

$$Z_0^{\parallel}(k) = \frac{Z_0}{2\pi^{3/2}b}[1+\mathrm{sgn}(k)i]\sqrt{\frac{g}{|k|}} \tag{12}$$

where $Z_0 = \frac{4\pi}{c} = 377\ \Omega$.

Our results offer an opportunity to check the diffraction model with exact Maxwell solutions. (Our result is not a rigorous proof of the diffraction model because we do not have a cylindrical geometry.) Consider a surface charge beam with $\Sigma(x,t) = \Sigma_0 e^{ik(x-ct)}$, which moves with the speed of light c. The wake fields can be obtained from the rod-beam results by superposition. Take the disrupted beam case with $\theta = 0$ for example. We have

$$\begin{aligned} B_z &= -\frac{4\cos\frac{\phi}{2}}{\sqrt{2r}}\Sigma_0 e^{ik(r-ct)}\int_0^\infty du\frac{e^{iku}}{\sqrt{u}(\frac{u}{r}+1+\cos\phi)} \\ E_y &= \frac{4\cos\frac{\phi}{2}}{\sqrt{2r}}\Sigma_0 e^{ik(r-ct)}\int_0^\infty du\frac{e^{iku}(\frac{u}{r}+\cos\phi)}{\sqrt{u}(\frac{u}{r}+1+\cos\phi)} \end{aligned} \tag{13}$$

Significant contributions to the integrals (13) come from the region $u < \frac{1}{|k|}$. This in turn means that the components B_z and E_y are strong when ϕ is close to π with

$$|\pi-\phi| < \sqrt{\frac{2}{|k|r}} \tag{14}$$

Equation (14) in turn gives the diffraction pattern (11).

The component E_x, however, is somewhat different. It does not have the diffraction pattern (11). In fact,

$$E_x = -\frac{4\sin\frac{\phi}{2}}{\sqrt{2r}}\Sigma_0 e^{ik(r-ct)}\int_0^\infty du\frac{e^{iku}}{\sqrt{u}}$$
$$= -\frac{4\sin\frac{\phi}{2}}{\sqrt{2r}}\Sigma_0 e^{ik(r-ct)}\sqrt{\frac{\pi}{2|k|}}[1+\mathrm{sgn}(k)i] \qquad (15)$$

The magnitude of E_x however is smaller than those of B_z and E_y by a factor of $|k|r \gg 1$.

One can estimate the high frequency impedance as follows. Consider a test charge which passes position $x = -D$ at time $t = 0$ with a vertical separation Y from the wedge. Assume the test charge move in the x-direction at the speed of light. The energy loss of the test charge as it traverses the cavity can be estimated as (assume $g \gg D, g \gg Y, |k|g \gg 1$)

$$\Delta\mathcal{E} \approx \int_0^g d(ct)eE_x \approx -4\sqrt{\frac{\pi g}{|k|}}[1+\mathrm{sgn}(k)i]e\Sigma_0 e^{-ikD} \qquad (16)$$

Although (16) is for a geometry with infinite z-dimension, the impedance of a cylindrical cavity can be estimated by

$$Z_0^{\parallel}(k) = \frac{\Delta\mathcal{E}/e}{2\pi bc\Sigma_0 e^{-ikD}} \qquad (17)$$

which is identical to (12). One can show that (12) applies also to arbitrary θ. The diffraction model is therefore re-established. Further exploring of more details of the diffraction model should be possible using the exact solutions given in the previous sections.

VI. EMITTANCE GROWTH

We now estimate the emittance growth when a flat beam is being collimated by a metal collimator. Let the horizontal distribution of the beam be uniform with a total width L_z. We assume the vertical beam dimension is $\ll L_z$, and it is the vertical dimension which is being collimated. The vertical separation between the flat beam and the edge of the collimator is assumed to be $\ll L_z$. We ignore the resistive wall effect here.[6,7]

Consider the case of a semi-infinite screen wedge. Let the surface charge density of the beam be written as $\Sigma(x) = \frac{Ne}{L_z}\rho(x)$, where N is the total number of particles in the beam bunch, and $\int_{-\infty}^{\infty} dx\,\rho(x) = 1$. The kick angle received by a particle in the beam located at longitudinal position x is, according to Eq.(9),

$$\Delta y'(x) = \frac{\pi N r_0}{L_z\gamma}\int_x^\infty dx'\rho(x') \qquad (18)$$

where r_0 is the classical radius of the particle, γ is the Lorentz energy factor.

The maximum kick is received by particles in the trailing tail $x = -\infty$. Independent of the details of the longitudinal distribution $\rho(x)$, this kick is given by

$$\Phi \equiv \Delta y'(-\infty) = \frac{\pi N r_0}{L_z\gamma} \qquad (19)$$

The growth in the effective emittance of the beam is also independent of the details of $\rho(x)$:

$$\Delta\epsilon = \int_{-\infty}^{\infty} dx\,\rho(x)\beta\Delta y'^2(x) = \frac{1}{3}\beta\Phi^2 \qquad (20)$$

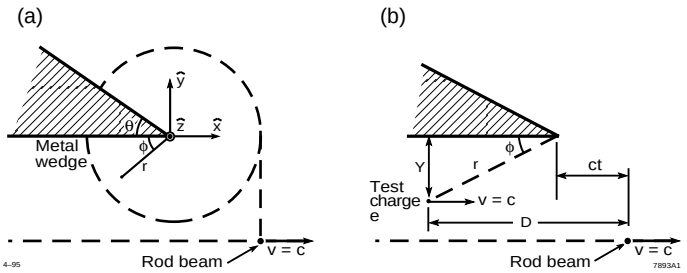

Figure 1. A rod beam passing a disrupted pipe wedge.

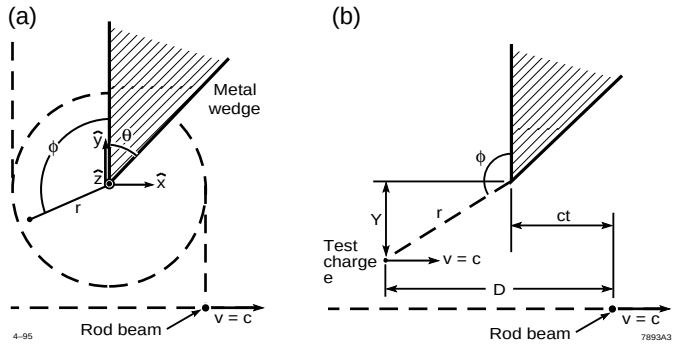

Figure 2. A rod beam passing a semi-infinite screen wedge.

where β is the β-function at the collimator.

As a numerical example, take an electron beam bunch with $N = 5\times10^{10}$, $L_z = 1$ mm, and $\gamma = 10^5$. We assume that the vertical beam height and the vertical distance of the beam from the collimator are $\ll 1$ mm. If we further assume the collimator has a semi-infinite screen geometry, then the wake field kick delivered to a trailing particle in the bunch is 4 μrad. If $\beta = 10$ m, the effective emittance growth is found to be 0.6×10^{-10} m-rad, which corresponds to a growth of normalized emittance of 0.6×10^{-5} m-rad. As mentioned in Section 1, this can be detrimental for a high quality, low-emittance beam.

References

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