ON THE RELAXATION OF SEMI-GAUSSIAN AND K-V BEAMS TO THERMAL EQUILIBRIUM *

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Abstract

A beam propagating in a continuous, linear focusing channel tends to relax to a thermal equilibrium state. We employ nonlinear conservation constraints to theoretically analyze changes in quantities that characterize both an initial semi-Gaussian beam with a matched rms beam envelope and a K-V beam under a relaxation to thermal equilibrium. Results from particle-in-cell simulations are compared to the theoretical predictions.

I. INTRODUCTION

Semi-Gaussian (SG) beams are characterized by a thermal-like Gaussian distribution of particle momentum and uniformly distributed space-charge. In so-called K-V beams first described by Kapchinskij and Vladimirskij, all particles have the same transverse energy and the space-charge is also uniformly distributed.[1,2] Both SG and K-V beams are widely used in the theory and simulation of charged particle beams, and a fundamental question is how these beams change on relaxation to thermal equilibrium (TE). Here we employ conservation constraints of a simple theoretical model to derive equations that connect initial SG and K-V beams to their final TE state. These equations are solved numerically to obtain universal curves describing changes in beam emittance, radius, and peak density on relaxation to TE. These curves demonstrate contexts in which these distributions may be regarded as approximations to TE. This study does not address the dynamical evolution of the beam as it relaxes to TE.

II. THEORETICAL MODEL, MOMENTS, AND CONSERVATION CONSTRAINTS

We employ an (r, θ, z) cylindrical polar coordinate system to analyze an infinitely long, unbunched ($\partial/\partial z = 0$) beam composed of a single species of particles of mass m and charge q. All particles propagate with constant axial velocity $v_b \mathbf{e}_z$, and continuous radial focusing is provided by an external electric field that is proportional to the radial coordinate r, i.e., $\mathbf{E}_{\rm ext} = -(mv_b^2 k_\beta^2/q) r \mathbf{e}_r$, where $k_\beta =$ const is the betatron wavenumber. This field can be thought of as arising from a uniform background of charges or as representing the average focusing strength of an alternating gradient lattice of electric or magnetic quadrupoles.[1,2] For simplicity, we neglect self-magnetic fields and employ a nonrelativistic and electrostatic model where initial ($s = 0$) and final ($s \to \infty$) states of the beam can be described for a long axial propagation distance s ($s = v_b t$, where t is the time) in terms of a single-particle distribution function f that can generally be a function of the transverse position and momentum $\mathbf{x}$ and $\mathbf{p}$ of a single particle and the axial coordinate s, i.e., $f = f(\mathbf{x}, \mathbf{p}, s)$. Neglecting particle correlation effects, the evolution of f is described by the Vlasov equation,[1]

$$\left\{ \frac{\partial}{\partial s} + \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{x}} - \frac{\partial H}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{p}} \right\} f(\mathbf{x}, \mathbf{p}, s) = 0, \qquad (1)$$

where

$$H(\mathbf{x}, \mathbf{p}) = \frac{\mathbf{p}^2}{2mv_b} + mv_b \frac{k_\beta^2}{2} \mathbf{x}^2 + \frac{q}{v_b} \phi \qquad (2)$$

is the Hamiltonian and the self-field potential ϕ satisfies the Poisson equation

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] \phi(\mathbf{x}, s) = -4\pi q \int d^2p \, f. \qquad (3)$$

Beam Vlasov equilibria are stationary ($\partial/\partial s = 0$) solutions to the Vlasov-Poisson system (1)-(3). It follows that any distribution function f formed from the single-particle constants of the motion in the full equilibrium field configuration is a Vlasov equilibrium. Therefore, for azimuthally symmetric ($\partial/\partial\theta = 0$) beams, $f = F(H)$ is an equilibrium distribution for arbitrary functions $F(H)$. It can be shown that the equilibrium $f = F(H)$ is stable to perturbations of arbitrary amplitude if $dF(H)/dH \leq 0$ [1]. Moreover, the density inversion theorem[1] shows that any beam equilibrium with a radial density profile $n(r) = \int d^2p \; f$ satisfying $dn/dr \leq 0$ corresponds to a stable distribution $f = F(H)$ with $dF(H)/dH \leq 0$.

Moment descriptions of the beam can provide a simplified understanding of beam transport. Transverse statistical averages of a quantity ξ are expressed in terms of this Vlasov formulation as $\langle \xi \rangle \equiv (1/N) \int d^2x \int d^2p \; \xi \; f$, where $N \equiv \int d^2x \int d^2p \; f$ is the number of particles per unit axial length. A commonly employed measure of the envelope radius of beam particles is the rms radius $R \equiv \sqrt{2\langle r^2 \rangle}$. Note that R is identically equal to the edge radius of a beam with uniformly distributed space-charge. Second order moments of the Vlasov equation (1) can be employed to derive the so-called "rms envelope equation" for the evolution of R.[1,2] For azimuthally symmetric beams (i.e., $\partial/\partial\theta = 0$), one obtains

$$\frac{d^2R}{ds^2} + k_\beta^2 R - \frac{K}{R} - \frac{\epsilon_x^2}{R^3} = 0 \; , \qquad (4)$$

where $K = -2q\langle r\partial\phi/\partial r \rangle / mv_b^2$ is the self-field perveance [Eq. (3) can be integrated to obtain $\langle r\partial\phi/\partial r \rangle = -qN$, and thereby show that $K = 2q^2N/mv_b^2 =$ const] and

$$\epsilon_x^2 = 16[\langle x^2 \rangle \langle (dx/ds)^2 \rangle - \langle x(dx/ds) \rangle^2] \qquad (5)$$

is the square of the rms x-emittance ϵ_x. For a K-V equilibrium distribution, $\pi\epsilon_x$ is constant and corresponds to the phase-space

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area (in x, dx/ds phase-space) of the beam. For general distributions, ϵ_x is not constant and is employed as a statistical measure of the quality of the beam.[2]

It is convenient to express the envelope equation (4) as $d^2R/ds^2 + \sigma^2 R - \epsilon_x^2/R^3 = 0$, where $\sigma \equiv (k_\beta^2 - K/2\langle r^2\rangle)^{1/2}$ is the phase-advance per unit axial length of the transverse oscillations of a single particle moving in the applied and self-fields of an "equivalent" K-V beam.[2] For radial confinement of the beam ($n = \int d^2p\ f\ = 0$ in the limit $r \to \infty$), $\sigma^2 \geq 0$, with the limit $\sigma = 0$ corresponding to a cold-beam equilibrium with $dR/ds = 0 = d^2R/ds^2$ and $\epsilon_x^2 = 0$. In the tenuous, kinetic-dominated limit $k_\beta^2 \gg K/2\langle r^2\rangle$, space-charge effects are negligible, and $\sigma \simeq \sigma_0$, where $\sigma_0 = |k_\beta|$ is the "undepressed" phase advance. The phase advance σ in the presence of space-charge is "depressed" from σ_0 (i.e., $\sigma^2 = \sigma_0^2 - K/2\langle r^2\rangle$), and the phase advance ratio

$$\sigma/\sigma_0 = (1 - K/2k_\beta^2\langle r^2\rangle)^{1/2} \tag{6}$$

provides a convenient normalized measure of space-charge effects ($0 \leq \sigma/\sigma_0 \leq 1$), with the limits $\sigma/\sigma_0 \to 0$ and $\sigma/\sigma_0 \to 1$ corresponding to a cold, space-charge dominated beam and a warm, kinetic dominated beam, respectively.

The nonlinear Vlasov-Poisson system (1)-(3) possesses the conservation constraints

$$N = \int d^2x \int d^2p\ f\ = \text{const}, \tag{7}$$

$$\mathcal{E} = \int d^2x \int d^2p\ \frac{\mathbf{p}^2}{2m} f\ +\ N m v_b^2 \frac{k_\beta^2}{2}\langle r^2\rangle\ +\ W\ =\ \text{const},$$

where $W \equiv \int d^2x\ |\partial\phi/\partial\mathbf{x}|^2/8\pi$ is the self-field energy. It can be verified that $dN/ds = 0 = d\mathcal{E}/ds$ follow directly from Eqs. (1)-(3). These constraints correspond to the conservation per unit axial length of particle number and system energy (particle and field) and provide powerful constraints on the nonlinear evolution of the system. Similar constraints remain valid in systems where particle correlation effects are not negligible. Note that the two-dimensional self-field energy W is logarithmically divergent since $\partial\phi/\partial\mathbf{x} \sim -(2qN/r)\mathbf{e}_r$ for $r \gg R$. For practical applications, this divergence must be removed (regularized) in an s-invariant manner. For azimuthally symmetric ($\partial/\partial\theta = 0$) beams, the divergence can be isolated by examining the work required to assemble the beam from a large radius.[3] Subtracting this divergence from W, we obtain the regularized self-field energy

$$W_r = -8\pi^2 q^2 \int_0^\infty dr\ r \ln\left(\frac{r}{r_s}\right) n(r) \int_0^r d\bar{r}\ \bar{r} n(\bar{r}), \tag{8}$$

where $r_s = $ const is a scale radius and $n(r) = \int d^2p\ f$ is the radial density. Making the replacement $W \to W_r$ in the constraint $\mathcal{E} = $ const obtains the needed regularized energy constraint. Insofar as the same scale radius $r_s = $ const is applied, this regularized conservation constraint can be applied to connect two azimuthally symmetric states, even if the intervening states are *not* azimuthally symmetric.

III. BEAM THERMAL EQUILIBRIA

A beam thermal equilibrium (TE) is characterized by a radial density profile that becomes uniform in the limit of low temperature and Gaussian-like for high temperature. The single-particle distribution function describing a TE beam is[1,2]

$$f = \frac{n_0}{2\pi m T} \exp\left\{-\frac{v_b H}{T}\right\}. \tag{9}$$

Here, $n_0 = $ const is a characteristic density and $T = $ const is the thermodynamic temperature (energy units). Specification of the charge and energy of the beam macrostate fix the constants n_0 and T. The TE distribution is a special class of stable Vlasov equilibrium.[1] Within the weak coupling approximation ($q^2/n_0^{-1/3} \ll T$) any initial distribution function $f(\mathbf{x}, \mathbf{p}, s = 0)$, however complex, relaxes to the TE form of Eq. (9). This is true regardless of the details of the intervening evolution due to both collective and collisional processes. Even stable Vlasov equilibria must ultimately relax to TE form due to effects outside the Vlasov model. In this regard, TE can be regarded as the preferred equilibrium state of the system.

Employing the TE distribution (9), one obtains

$$\epsilon_x^2 = \frac{8T}{m v_b^2}\langle r^2\rangle, \mathcal{E}_r = NT + m v_b^2 \frac{k_\beta^2}{2}\langle r^2\rangle + W_r, \tag{10}$$

where $\mathcal{E}_r$ denotes the regularized system energy. The envelope equation (4) with $d^2R/ds^2 = 0$ and ϵ_x^2 calculated above then shows that

$$\langle r^2\rangle = \frac{2T + q^2 N}{m v_b^2 k_\beta^2}. \tag{11}$$

The TE density $n(r) = \int d^2p\ f$ needed to explicitly calculate $N = 2\pi\int_0^\infty dr\ \ rn(r)$ and W_r is nonlinear, and must, in general, be calculated numerically. For this purpose it is convenient to express the density as $n(r) = n_0 \exp(-\psi)$, where $\psi \equiv (1/T)[m v_b^2 k_\beta^2 r^2/2 + q\phi]$ satisfies the transformed Poisson equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\psi}{d\rho}\right) = 1 + \Delta - \exp(-\psi),$$

subject to $\psi(0) = 0$. Here, $\rho \equiv r/\lambda_D$ is a radial coordinate scaled to the thermal Debye length $\lambda_D \equiv (T/4\pi q^2 n_0)^{1/2}$ formed from the on-axis beam density n_0, and $1 + \Delta \equiv 2v_b^2 k_\beta^2/\omega_{p0}^2$ (where $\omega_{p0}^2 \equiv 4\pi q^2 n_0/m$ is the on-axis plasma frequency-squared), is a positive, dimensionless parameter qualitatively representing the ratio of applied to space-charge defocusing forces.

IV. SEMI-GAUSSIAN AND K-V BEAMS

Semi-Gaussian (SG) and K-V beams are described by the single-particle distribution functions[1−3]

$$f = \begin{cases} (n_b/2\pi m T_b)\Theta(r_b - r)\exp(-\mathbf{p}^2/2mT_b), & \text{SG} \\ (n_b/2\pi m v_b)\delta(H - 2T_b/v_b), & \text{K-V.} \end{cases} \tag{12}$$

Here, $\Theta(x)$ and $\delta(x)$ are theta- and Dirac delta-functions, $T_b = $ const is the beam kinetic temperature [i.e., $NT_b =$

$\int d^2x \int d^2p\, (\mathbf{p}^2/2m)f]$ for both distributions, and both density profiles $n = \int d^2p\, f$ have a constant value n_b within the beam radius $r_b = \text{const}$ [i.e., $n = n_b$ for $0 \leq r < r_b$] and are zero outside the beam radius [i.e., $n = 0$ for $r > r_b$]. K-V distributions are exact Vlasov equilibria with $\partial/\partial s = 0$, whereas SG distributions are not and will evolve within the Vlasov model. The SG distribution must be regarded as an initial state, and the conditions $dR/ds = 0 = d^2R/ds^2$ are applied to this state for a "matched" beam envelope. Then for both the SG and K-V distributions, the envelope equation (4) becomes an initial state constraint $2T_b = mv_b^2 k_\beta^2 r_b^2/2 - q^2\pi r_b^2 n_b/2$ [equivalent to Eq. (11) and defining r_b for K-V beams], and expressions identical in form are obtained to the TE quantities calculated in Eq. (10) with the kinetic temperature T_b replacing the thermodynamic temperature T. Additionally, for these rectangular density profile beams, one may analytically calculate $N = \pi r_b^2 n_b$, $\langle r^2 \rangle = r_b^2/2$, and $W_r = (q^2N^2/4)[1 - 4\ln(r_b/r_s)]$.

V. BEAM CHANGES ON RELAXATION TO THERMAL EQUILIBRIUM

The conservation constraints $N = \text{const}$ and $\mathcal{E}_r = \text{const}$ uniquely connect an initial SG beam with a matched envelope or an initial K-V beam to its final TE state. These constraints can be used with Eqs. (10) and (11) expressed in scaled form for both initial and final states to calculate ratios of final to initial state emittance-squared ϵ_x^2, mean-square radius $\langle r^2 \rangle$, and on-axis density $n(r = 0)$ in terms of the single dimensionless parameter Δ associated with the final TE. Likewise, the phase advance ratio σ/σ_0 of the initial beam can also be calculated in terms of Δ. Details of this procedure are presented elsewhere,[3] and the resulting ratios are plotted verses σ/σ_0 in the figure. These curves are universal in the sense that all beam parameters fall onto a single curve and the curves apply both to an initial (matched) SG or a K-V beam. Particle-in-cell simulations of an initial SG beam are presented in the figure. These simulations provide insight on the axial propagation distance necessary for relaxation to TE and agree well with the theory for σ/σ_0 small.[3,4] Spreads about the simulation points indicate rms fluctuations that become large as $\sigma/\sigma_0 \to 1$ due to the lack of space charge forces to induce relaxation to TE. Indeed, it has been shown that the simulations are consistent with relaxation to a virtual, phase-mixed equilibrium as $\sigma/\sigma_0 \to 1$.

The figure shows that the rms emittance and radius undergo small, space-charge dependent decreases on relaxation to TE, while the peak (on-axis) beam density can undergo a significant, space-charge dependent increase. All the ratios plotted approach unity in the space-charge dominated limit $\sigma/\sigma_0 \to 0$ because all three distributions become identical with uniform densities and zero temperatures in this limit. In the kinetic dominated limit $\sigma/\sigma_0 \to 1$, the ratios of rms emittance and mean-square radius approach unity, whereas the ratio of peak densities approaches 2. These limits are consistent with analytic calculations with the self-field potential ϕ neglected. With ϕ neglected, all second-order moments of the system are constants of the motion (the rms radius and temperature are then constants) and the final TE density profile is Gaussian with $n(r) = n_0 \exp(-2r^2/r_b^2)$, thereby showing that the ratio of final to initial peak density is 2 using

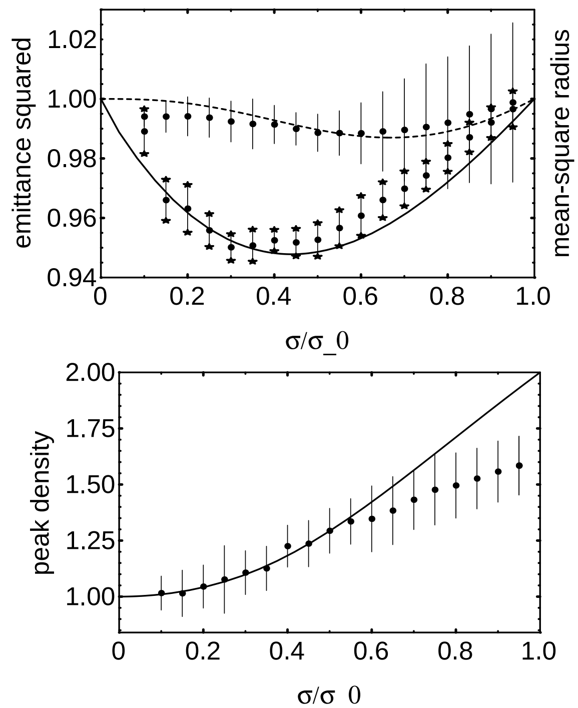

Figure. 1. Ratios of final to initial state emittance squared (ϵ_x^2, upper plot, solid curve), mean-square radius ($\langle r^2 \rangle$, upper plot, dashed curve), and peak (on-axis) density [$n(r = 0)$, lower plot] verses σ/σ_0.

$N = \pi r_b^2 n_b = \int d^2x\; n(r)$. For the general case of finite space-charge effects with $0 < \sigma/\sigma_0 < 1$, the figure demonstrates that on relaxation to TE, the rms radius of the beam decreases slightly, while the peak (on-axis) beam density significantly increases. Evidently, the initially uniform density beam relaxes to a diffuse radial density profile such that the characteristic thermal tail and increased core density weight to maintain nearly constant rms radius.

VI. CONCLUSIONS

An initial semi-Gaussian or K-V beam within a continuous focusing channel must ultimately relax to thermal equilibrium. We employed conservation constraints of a simple theoretical model to analyze changes in quantities characterizing the beam under this relaxation. Universal curves were calculated giving the ratio of various final to initial state quantities in terms of the ratio of depressed to undepressed phase advance of the initial beam, which provides a convenient normalized measure of space-charge effects. These curves demonstrate that the rms emittance and radius of the beam undergo a small, space-charge dependent decrease on relaxation to TE. The smallness of these decreases for σ/σ_0 small indicate that with respect to the transport of second-order moments of the system, which are of primary importance in beam physics, the SG and K-V distributions are a good approximation to the true TE distribution that can be transported without change. On the other hand, particularly for larger σ/σ_0, it was demonstrated that higher order moments or nonmoment quantities (e.g., peak beam density) could undergo significant space-charge dependent changes on relaxation to TE, thereby indicating both contexts for caution and possible measures to ascertain whether the beam has relaxed. More detailed analyses including beam rotation and magnetic focusing along with relativistic, self-magnetic, and longitudinal effects have been carried out, and the essential conclusions of this simple analysis remain unaltered.[3]

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