FREQUENCY DEPENDENCE OF THE POLARIZABILITY AND SUSCEPTIBILITY OF A CIRCULAR HOLE IN A THICK CONDUCTING WALL*

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Abstract

We calculate a generalized polarizability and susceptibility of a circular hole in a thick metallic plate as a function of hole dimensions and wavelength. In particular, we construct a variational form which allows us to obtain accurate numerical results with a minimum of computational effort. Numerical results are obtained for a variety of hole dimensions relative to the wavelength. In addition, analytic results are obtained and shown to be accurate to second order in the ratio of the hole dimension to the wavelength for a vanishingly thin wall.

I. INTRODUCTION

The penetration of electric and magnetic fields through a hole in a metallic wall plays an important role in many devices. In an accelerator, such holes in the beam pipe serve to allow access for pumping, devices for beam current and beam position measurement, coupling between cavities, etc. In much of the early work the hole dimensions were considered to be very small compared to the wavelength. The purpose of this paper is to extend the calculation to include the effects of finite wavelength, although we still confine our attention to wavelengths no smaller than the hole dimensions.

We redefine the conventional static treatment of polarizability and susceptibility in terms of the cavity detuning, defining a new generalized polarizability and susceptibility. In this way, we include the frequency dependence of the polarizability and susceptibility as well as the contributions of higher multipole moments of the hole. But these generalized polarizability and susceptibilites should only be seen as intermediate vehicles to relate the coupling integrals of interest to the detuning of the cavity by the hole. We will obtain an expression for the detuning of the modes of the symmetric cavity structure due to the presence of the hole. The symmetric cavity structure consists of two identical cavities, each of the length L and radius b. Clearly the modes will be either symmetric or antisymmetric in the axial coordinate. Our analysis will be limited to the modes near the $\mathrm{TM}_{0N\ell}$, $\mathrm{TM}_{1N\ell}$ and $\mathrm{TE}_{1N\ell}$ modes of the unperturbed pillbox.

II. GENERAL ANALYSIS

Our analysis can be generalized to include cavity regions and iris regions of arbitrary cross section. Taking $z_1 = 0$ to be the left end of the left cavity, we can write the transverse electric field as

$$\boldsymbol{E}_\perp^{(C)}(\boldsymbol{r}, z_1) = \sum_n a_n \boldsymbol{e}_n(\boldsymbol{r}) \frac{\sin\beta_n z_1}{\sin\beta_n L}, \tag{1}$$

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where $\boldsymbol{r}$ stands for the transverse coordinates x and y, and where the modes $\boldsymbol{e}_n$ may be either TM or TE. Here $\beta_n^2 = k^2 - \gamma_n^2$, where γ_n^2 are the eigenvalues of the two dimensional scalar Helmholtz equation in the cavity region with the appropriate boundary conditions. We use Latin subscripts $(n, m, N, \cdots)$ for the cavity region, and $kc/2\pi$ is the frequency.

The transverse electric field in the iris region can similarly be written as

$$\boldsymbol{E}_\perp^{(I)}(\boldsymbol{r}, z_2) = \sum_\nu b_\nu \boldsymbol{e}_\nu(\boldsymbol{r}) \frac{\cos\beta_\nu z_2}{\cos\beta_\nu g/2}, \tag{2}$$

where $z_2 = 0$ is now the center of the iris region, and where we use Greek subscripts $(\nu, \mu, \sigma, \cdots)$ for the iris region. Equation (2) is appropriate for the modes in our symmetric structure for which $\boldsymbol{E}_\perp^{(I)}$ is even in z_2. For those modes where $\boldsymbol{E}_\perp^{(I)}$ is odd in z_2 we need to replace the cosines by sines in Eq. (2). We express the coefficients a_n and b_ν in terms of $\boldsymbol{E}_\perp(\boldsymbol{r}) \equiv \boldsymbol{u}(\boldsymbol{r})$ at the interface between the cavity and the iris $(z_1 = L,\ z_2 = -g/2)$. We then write the transverse magnetic field in each region. Equating the transverse magnetic field at $z_1 = L, z_2 = -g/2$ in the region S_1 leads to the integral equation for the unknown function $\boldsymbol{u}(\boldsymbol{r})$

$$\int_{S_1} dS'\ \boldsymbol{u}(\boldsymbol{r}') \cdot \overleftrightarrow{K}(\boldsymbol{r}, \boldsymbol{r}') = 0, \tag{3}$$

where

$$\begin{aligned} \overleftrightarrow{K}(\boldsymbol{r}, \boldsymbol{r}') &= \sum_n \lambda_n \boldsymbol{e}_n(\boldsymbol{r}) \boldsymbol{e}_n(\boldsymbol{r}') \cot\beta_n L \\ &- \sum_\nu \lambda_\nu \boldsymbol{e}_\nu(\boldsymbol{r}) \boldsymbol{e}_\nu(\boldsymbol{r}') \tan\beta_\nu g/2, \end{aligned} \tag{4}$$

and

$$\lambda_n = \frac{Z_0}{Z_n} = \left\{ \begin{array}{ll} k/\beta_n\ , & TM \\ \beta_n/k\ , & TE \end{array} \right\},$$

$$\lambda_\nu = \frac{Z_0}{Z_\nu} = \left\{ \begin{array}{ll} k/\beta_\nu\ , & TM \\ \beta_\nu/k\ , & TE \end{array} \right\}, \tag{5}$$

with $Z_0 = \sqrt{\mu_0/\epsilon_0}$ being the impedance of free space, Z_n being the impedance of the "cavity" wave guide and Z_ν is the impedance of the "iris" waveguide.

By separating out the dominant term N in the sum over n, one can cast Eq. (3) into a variational form for the frequency, with the trial function being $\boldsymbol{u}(\boldsymbol{r})$. The result is the implicit equation for the frequency

$$\frac{\tan\beta_N L}{\lambda_N} = \sum_\nu \sum_\mu K_{N\nu} (\boldsymbol{M}^{-1})_{\nu\mu} K_{N\mu}. \tag{6}$$

Here $M_{\nu\mu}$ is a symmetric matrix defined by

$$M_{\nu\mu} = - \sum_{n\neq N}{}' \lambda_n \cot\beta_n L\; K_{n\mu}K_{n\nu} + \sum_\sigma \lambda_\sigma \tan\beta_\sigma g/2\; K_{\sigma F\mu}K_{\sigma\nu} \qquad (7)$$

and

$$K_{n\nu} \equiv \int_{S_1} dS\; \boldsymbol{e}_n \cdot \boldsymbol{f}_\nu\;,\; K_{\sigma\nu} \equiv \int dS\; \boldsymbol{e}_\sigma \cdot \boldsymbol{f}_\nu. \qquad (8)$$

Note that we are looking at modes close to the cavity modes corresponding to $\beta_N L = \ell\pi$ or $k^2_{N\ell} = \ell^2\pi^2/L^2 + \gamma^2_N$. Since the frequency k is also contained in $M_{\nu\mu}$, Eq. (6) must be solved by iteration.

We obtain the expression for the frequency change in the cavity and therefore generalize the concept of the static polarizability to a generalized polarizability by considering a mode which has a normal electric field, but no tangential magnetic field at the center of the hole (for example the $TM_{0N\ell}$ mode) and write

$$\chi \equiv \frac{k^2 - k^2_M}{k^2_M E^2_M(0)}. \qquad (9)$$

Similarly we obtain the generalization of the susceptibility for a circular hole (for example the $TM_{1N\ell}$ or $TE_{1N\ell}$ mode)

$$\psi \equiv \frac{k^2_M - k^2}{k^2 H^2_M(0)}. \qquad (10)$$

We obtain k from Eq. (6) and χ or ψ , for that frequency, from Eq. (9) or (10). Clearly the mode identification M stands for $0N\ell$ or $1N\ell$ as appropriate.

III. POLARIZABILITY FOR A CIRCULAR IRIS HOLE

We now specialize to TM_{0n} waveguide modes in a circular geometry in order to obtain the polarizability[1]. The cavity radius is b and the iris radius is a and we use the complete set

$$\boldsymbol{f}_\nu(\boldsymbol{r}) = \boldsymbol{e}_\nu(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi_\nu(r), \qquad (11)$$

with

$$\phi_n(r) = \frac{J_0(s_n r/b)}{\sqrt{\pi} s_n J_1(s_n)},\; \phi_\nu(r) = \frac{J_0(s_\nu r/a)}{\sqrt{\pi} s_\nu J_1(s_\nu)}. \qquad (12)$$

Here $s_{n,\nu}$ are the roots of the equation $J_0(s_{n,\nu}) = 0$. Evaluating the kernels and matrices in Eq. (8) and Eq. (7), the frequency can now be calculated using Eq. (6) and the symmetric polarizability using Eq. (9). Similarily we can obtain the asymmetric polarizability.

The polarizabilities obtained in this way will be functions of the geometrical parameters $a/b, g/a, a/L$, and ℓ, N of the $TM_{0N\ell}$ cavity mode. In order to tie the polarizabilities to the geometry of the hole alone, it is necessary to take the limit for large b and L, but with finite frequency. This can be accomplished by letting $b, L \to \infty$, but keeping $s \equiv s_N a/b \quad t \equiv \ell\pi a/L$ finite by also allowing $N, \ell \to \infty$.

IV. SUSCEPTIBILITY FOR A CIRCULAR HOLE

We now must use the waveguide modes TM_{1n} and TE_{1n} for our complete set in the pipe region. Specifically we have

$$\boldsymbol{e}_n = -\boldsymbol{\nabla}\phi_n,$$

$$\phi_n = \sqrt{\frac{2}{\pi}}\frac{J_1(p_n r/b)\cos\theta}{p_n J_0(p_n)} \text{ for } \mathrm{TM_{1n}} \text{ modes}, \qquad (13)$$

$$\boldsymbol{e}_n = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi_n,$$

$$\psi_n = \sqrt{\frac{2}{\pi}}\frac{J_1(q_n r/b)\sin\theta}{\sqrt{q_n^2-1}J_1(q_n)} \text{ for } \mathrm{TE_{1n}} \text{ modes}, \qquad (14)$$

where $p_{n,\nu}$ are the roots of $J_1(p_{n,\nu}) = 0$. In the iris region

$$\boldsymbol{e}_\nu = -\nabla\phi_\nu,$$

$$\phi_\nu = \sqrt{\frac{2}{\pi}}\frac{J_1(p_\nu r/a)\cos\theta}{p_\nu J_0(p_\nu)} \text{ for } \mathrm{TM}_{1\nu} \text{ modes}, \qquad (15)$$

$$\boldsymbol{e}_\nu = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi_\nu,$$

$$\psi_\nu = \sqrt{\frac{2}{\pi}}\frac{J_1(q_\nu r/a)\sin\theta}{\sqrt{q_\nu^2-1}J_1(q_\nu)} \text{ for } \mathrm{TE}_{1\nu} \text{ modes}, \qquad (16)$$

where $q_{n,\nu}$ are the roots of $J'_1(q_{n,\nu}) = 0$. Once again, the kernels and matrix elements can be calculated, but now n and ν can be either TM or TE in $K^\psi_{n\nu}$, where the superscript ψ denotes susceptibility. The frequency is again obtained from Eq. (6), but the susceptibility requires using Eq. (10).

V. DISCUSSION OF THE ANALYTIC RESULTS

The variational formulation also allows us to obtain analytically the first order correction in k^2a^2 for a hole in a plate of zero thickness from the knowledge of the correct field solutions for $ka = 0$. In the case of polarizbility we obtain the following approximate analytic form

$$\chi_{N\ell} \cong \frac{4a^3}{3}\left(1 - \frac{k^2a^2}{5} - \frac{s^2}{5} + \frac{4j}{9\pi}k^3a^3\right). \qquad (17)$$

In the case of susceptibility we obtain

$$\psi^{TM}_{N\ell} \cong \frac{8a^3}{3}\left[1 + k^2a^2\left(\frac{8}{15} - \frac{p^2}{3k^2a^2} - \frac{p^4}{15k^4a^4}\right) - \frac{8j}{9\pi}k^3a^3\right], \qquad (18)$$

$$\psi^{TE}_{N\ell} \cong \frac{8a^3}{3}\left[1 + k^2a^2\left(\frac{8}{15} - \frac{q^2}{5k^2a^2}\right) - \frac{8j}{9\pi}k^3a^3\right], \qquad (19)$$

where $p \equiv p_N a/b,\; q = q_N a/b$. It appears that the approximate forms for polarizability and susceptibility give reasonably accurate results even for values of ka as large as 1.

The coefficients of the leading terms in Eqs. (17)-(19) for $g = 0$ and small ka appear to be well confirmed. Moreover, the coefficients of the terms in first order in k^2a^2 and the coefficients of the leading imaginary terms are the same as those obtained by Eggimann[2] .

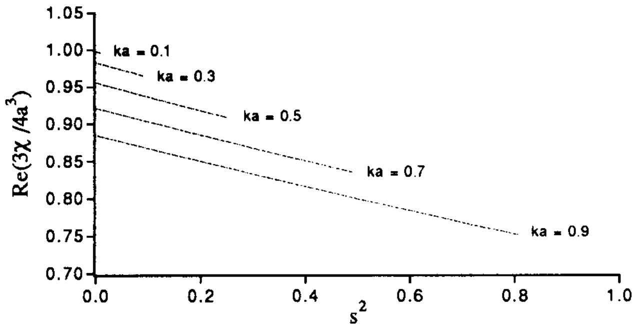

Figure. 1. Real part of scaled polarizability $\tilde{\chi}$ vs. s^2 for various ka, with $g/a = 0$.

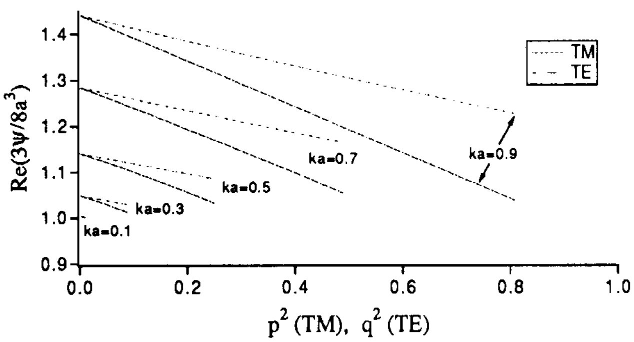

Figure. 2. Real part of scaled susceptibility $\tilde{\psi}$ vs. p^2 (for TM mode) or q^2 (for TE mode) for various ka, with $g/a = 0$.

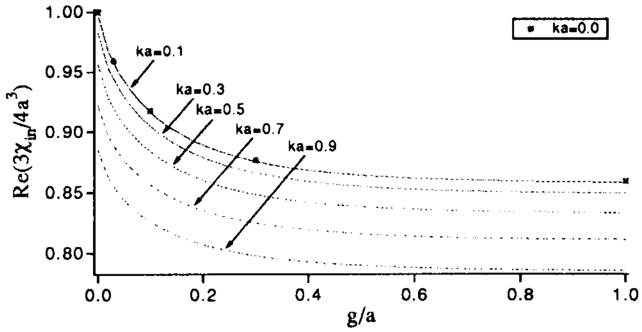

Figure. 3. Real part of scaled polarizability $\tilde{\chi_{in}}$ vs. g/a for various ka, with $s = 0$.

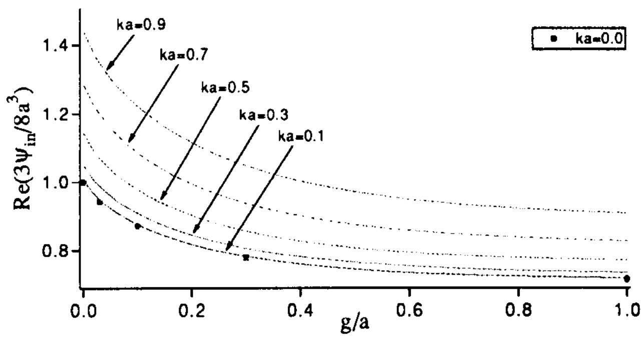

Figure. 4. Real part of scaled susceptibility $\tilde{\psi_{in}}$ vs. g/a for various ka, with $p = 0$ (for TM mode) or $q = 0$ (for TE mode), where the curves are the same for both TM and TE modes.

VI. NUMERICAL RESULTS FOR A HOLE IN A PLATE

We present the results in a form suggested by the predictions for $g = 0$ (zero wall thickness) and small ka, s in Eq. (17). Specifically, we define $\tilde{\chi}(ka, s) \equiv \chi/(4a^3/3)$ and plot the real part of $\tilde{\chi}$ vs. s^2 for various values of ka in Fig. 1. A similar presentation is provided for the real part of the susceptibilities. We define $\tilde{\psi} \equiv \psi/(8a^3/3)$ and plot $\tilde{\psi}$ vs. p^2 (TM) or q^2 (TE) for various values of ka in Fig. 2.

For the wall with finite thickness, the polarizability and susceptibility seen within the cavity are given by[1] $\chi_{in} = \chi_s + \chi_a$, $\psi_{in} = \psi_s + \psi_a$, while the polarizability and susceptibility seen outside the cavity are given by $\chi_{out} = \chi_s - \chi_a$, $\psi_{out} = \psi_s - \psi_a$. Here the subscripts s and a denote the solutions of the symmetric and antisymmetric potential problems[1] . In Figs. 3 and 4, we show $\tilde{\chi}_{in}$ and $\tilde{\psi}_{in}$ as functions of g/a for various ka.

The logarithmic plots of $\tilde{\chi}_{out}$ and $\tilde{\psi}_{out}$ become linear with slopes $-s_1 = -2.405$ and $-q_1 = -1.841$, respectively, as expected.

VII. SUMMARY

We have defined a generalized polarizability and susceptibility of a hole for finite wave length in terms of the frequency shift of the associated cavities due to the hole. In addition we have constructed a variational form for these frequency shifts, assuring good convergence for our numerical calculations. We then allow the cavity dimensions to be infinitely large, enabling us to obtain accurate numerical values of the polarizability and susceptibilities of a circular hole in an infinite plate of finite thickness. Then we obtain numerical results for various values of ka, g/a. The approximate analytic forms are given by Eq. (17) for the polarizability and by Eqs. (18), (19) for the susceptibility.

References

[1] The analysis for a circular hole in a thick wall follows that for $k = 0$ in R.L. Gluckstern and J.A. Diamond, IEEE Transactions in Microwave Theory and Techniques, Vo. 39, No. 2, February 1991, p. 274, which contains references to earlier work.

[2] W.H. Eggimann, Higher Order Evaluation of Electromagnetic Diffraction by Circular Disks, IRE Transactions of Microwave Theory and Techniques, vol. MMT-9, pp. 408-418, September 1961.