EMITTANCE GROWTH FROM ROTATED QUADRUPOLES IN HEAVY ION ACCELERATORS *

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Abstract

We derive a set of moment equations which incorporates linear quadrupolar focusing and space-charge defocusing, in the presence of rotational misalignments of the quadrupoles about the direction of beam propagation. Although the usual beam emittance measured relative to fixed transverse x and y coordinate axes is not constant, a conserved emittance-like quantity has been found. Implications for alignment tolerances in accelerators for heavy-ion inertial fusion are discussed.

I. INTRODUCTION

One class of misalignments of interest to accelerator designers is that class characterized by a rotation of the beam optical elements about the axis of propagation. Rotated dipoles, for example, are known to cause the centroid of a particle beam to wander off axis, (since the rotations will result in momentum impulses in the positive and negative y [vertical] direction.) However rotated quadrupoles will not cause an initially aligned beam centroid to become misaligned. Quadrupole rotations do however, create a linear coupling between the two transverse directions, x (horizontal) and y, in the equations of motion. Since this coupling enters linearly in the equations of motion, individual particle oscillation frequencies can be shifted, and this has implications for resonance avoidance in synchrotrons, (see ref. [1] and references therein). In this paper, we are interested in the effects on the emittance of beams with non-negligible space charge, such as those proposed for heavy ion inertial fusion.

We derive a set of moment equations which incorporates this coupling, and which serves as a generalization to the conventional envelope equations. We show that even when the equations of motion are linear in x and y, the beam emittance measured relative to fixed x and y coordinate axes is not constant, although a conserved emittance-like quantity can be defined. If not corrected, a beam will acquire a finite angular momentum and rotation angle, before passing through a final focusing lens, thereby limiting the achievable final spot size. The results presented here will be of use in determining alignment tolerances in heavy ion accelerators.

II. EQUATIONS OF MOTION

To obtain an estimate, we assume that the force on an ion comes from two sources only: The external focusing from a purely quadrupolar field, and the space charge of the beam (image forces have been neglected). For the purposes of this calculation we assume that the space charge is distributed in a uniform

density ellipse, but we allow the semi-axes and the rotation angle of the ellipse to evolve as a function of the axial coordinate z. We assume that a quadrupole is rotated by an angle θ from the x-axis, and that the beam is rotated by an angle α from the x-axis. The relation between the coordinates in the quadrupole frame (indicated by subscript 0) and the lab frame (no subscript) are given by:

$$x = x_0 \cos\theta - y_0 \sin\theta; \quad y = y_0 \cos\theta + x_0 \sin\theta \quad (1)$$

Similarly, the relation between the coordinates in the rotated beam frame (in which the beam semi-axes are parallel to the coordinate axes and are indicated by subscript b) and the lab frame are given by:

$$x - \langle x \rangle = x_b \cos\alpha - y_b \sin\alpha; \quad y - \langle y \rangle = y_b \cos\alpha + x_b \sin\alpha \quad (2)$$

Here $\langle \rangle$ indicates a statistical average over the distribution function. For a non-relativistic beam moving at constant velocity βc along the z axis, the paraxial equations of motion can then be written as:

$$x'' = K_{qxx}x + K_{qxy}y + K_{sxx}(x - \langle x \rangle) + K_{sxy}(y - \langle y \rangle) \quad (3)$$

$$y'' = K_{qyy}y + K_{qyx}x + K_{syy}(y - \langle y \rangle) + K_{syx}(x - \langle x \rangle) \quad (4)$$

Here primes $(')$ indicate derivatives with respect to z and10

$$K_{qxx} \equiv K_{qx0}\cos^2\theta + K_{qy0}\sin^2\theta = K_{qx0}\cos 2\theta$$
$$K_{qxy} \equiv (K_{qx0} - K_{qy0})(\sin\theta\cos\theta) = K_{qx0}\sin 2\theta$$
$$K_{qyx} \equiv (K_{qx0} - K_{qy0})(\sin\theta\cos\theta) = K_{qxy}$$
$$K_{qyy} \equiv K_{qy0}\cos^2\theta + K_{qx0}\sin^2\theta = -K_{qx0}\cos 2\theta$$
$$K_{sxx} \equiv K_{sxb}\cos^2\alpha + K_{syb}\sin^2\alpha$$
$$K_{sxy} \equiv (K_{sxb} - K_{syb})(\sin\alpha\cos\alpha)$$
$$K_{syx} \equiv (K_{sxb} - K_{syb})(\sin\alpha\cos\alpha) = K_{sxy}$$
$$K_{syy} \equiv K_{syb}\cos^2\alpha + K_{sxb}\sin^2\alpha$$

and where

$$K_{qx0} \equiv \pm \frac{B'}{[B\rho]} \quad \text{or} \quad \frac{E'}{\beta c[B\rho]}; \quad K_{qy0} \equiv -K_{qx0} \quad (5)$$

$$K_{sxb} \equiv K/[2(\Delta x_b^2 + (\Delta x_b^2 \Delta y_b^2)^{1/2})];$$
$$K_{syb} \equiv K/[2(\Delta y_b^2 + (\Delta x_b^2 \Delta y_b^2)^{1/2})] \quad (6)$$

and Δx_b^2 and Δy_b^2 are the moments in the rotated beam frame:

$$\Delta x_b^2 = \Delta x^2 \cos^2\alpha + \Delta y^2 \sin^2\alpha + 2\Delta xy \cos\alpha\sin\alpha \quad (7)$$

$$\Delta y_b^2 = \Delta y^2 \cos^2\alpha + \Delta x^2 \sin^2\alpha - 2\Delta xy \cos\alpha\sin\alpha \quad (8)$$

Here, $K \equiv 2qI/(\beta^3 AI_o)$ is the perveance, q is the charge state of the ions, A is the atomic mass of the ions, β is the velocity of the ions in units of c, $I_o \equiv 4\pi\epsilon_0 m_p c^3/e$ is the proton characteristic current ($\cong$31 MA), I is the ion beam current, B' and E' are quadrupole magnetic or electric field gradients respectively, $[B\rho] \equiv Am_p\beta c/qe$ is the ion rigidity, m_p is the proton mass, e is the proton charge, ϵ_0 is the free space permittivity and the operator Δ is defined (as in ref. [2]) by $\Delta ab = \langle ab \rangle - \langle a \rangle \langle b \rangle$ (e.g. $\Delta x^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$), where $\langle \; \rangle$ indicates average over particles.

Note that the space-charge force, is just the force obtained from the potential of a uniform density ellipse (ref. [3]), but where the semi-axes a and b have been replaced by $2(\Delta x_b^2)^{1/2}$ and $2(\Delta y_b^2)^{1/2}$, respectively, and where the location of the centroid determines the zero point of the space-charge force.

The beam rotation angle α may be expressed in terms of second order moments. From eq. (2), $\Delta x^2 - \Delta y^2 = (\Delta x_b^2 - \Delta y_b^2)\cos 2\alpha$ and $\Delta xy = (1/2)(\Delta x_b^2 - \Delta y_b^2)\sin 2\alpha$, so that

$$\tan 2\alpha = \frac{2\Delta xy}{\Delta x^2 - \Delta y^2} \qquad (9)$$

In deriving eq. 9 we have used the fact that $\Delta xy_b = 0$.

III. MOMENT EQUATIONS

Let the distribution function f, be the number of particles dN per unit transverse phase space volume,

$$f(x, x', y, y', z) = \frac{dN}{dx dx' dy dy'}$$

The evolution of f is described by the Vlasov/Collisonless Boltzmann Equation:

$$\frac{\partial f}{\partial z} + x'\frac{\partial f}{\partial x} + x''\frac{\partial f}{\partial x'} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} = 0, \qquad (10)$$

where x'' and y'' are determined by the equations of motion (eqs. [3] and [4]).

The average of a variable ξ over the continuous distribution is given by:

$$\langle \xi \rangle (z) \equiv \frac{1}{N}\int\int\int\int \xi f(x, x', y, y', z)dx dx' dy dy'.$$

Using integration by parts, it is straightforward to calculate the evolution of the following second order moments:

$\frac{d\Delta x^2}{dz} = 2\Delta xx'$

$\frac{d\Delta xx'}{dz} = \Delta x'^2 + K_{xx}\Delta x^2 + K_{xy}\Delta xy$

$\frac{d\Delta x'^2}{dz} = 2K_{xx}\Delta xx' + 2K_{xy}\Delta x'y$

$\frac{d\Delta y^2}{dz} = 2\Delta yy'$

$\frac{d\Delta yy'}{dz} = \Delta y'^2 + K_{yy}\Delta y^2 + K_{yx}\Delta xy$

$\frac{d\Delta y'^2}{dz} = 2K_{yy}\Delta yy' + 2K_{yx}\Delta xy'$

$\frac{d\Delta xy}{dz} = \Delta x'y + \Delta xy'$

$\frac{d\Delta x'y}{dz} = \Delta x'y' + K_{xx}\Delta xy + K_{xy}\Delta y^2$

$\frac{d\Delta xy'}{dz} = \Delta x'y' + K_{yy}\Delta xy + K_{yx}\Delta x^2$

$\frac{d\Delta x'y'}{dz} = K_{xx}\Delta xy' + K_{xy}\Delta yy' +$

$K_{yy}\Delta x'y + K_{yx}\Delta xx' \qquad (11)$

Similarly the evolution of the first-order moments is given by:

$\frac{d\langle x \rangle}{dz} = \langle x' \rangle$

$\frac{d\langle y \rangle}{dz} = \langle y' \rangle$

$\frac{d\langle x' \rangle}{dz} = K_{qxx}\langle x \rangle + K_{qxy}\langle y \rangle$

$\frac{d\langle y' \rangle}{dz} = K_{qyy}\langle y \rangle + K_{qyx}\langle x \rangle \qquad 12)$

In eq. 11,

$K_{xx} \equiv K_{qxx} + K_{sxx}$

$K_{yy} \equiv K_{qyy} + K_{syy}$

$K_{xy} \equiv K_{qxy} + K_{sxy}$

$K_{yx} \equiv K_{qyx} + K_{syx} = K_{xy}.$

Note that the four equations for the first order moments depend only on first order moments, and the ten equations for the second order moments depend only on second order moments, so that each set of equations forms a closed set. The centroid motion is thus decoupled from the envelope motion calculated with respect to the centroid.

The rms emittances defined along the x and y lab frame axes are defined by:

$\epsilon_x \equiv 4\left(\Delta x^2\Delta x'^2 - (\Delta xx')^2\right)^{1/2}$

$\epsilon_y \equiv 4\left(\Delta y^2\Delta y'^2 - (\Delta yy')^2\right)^{1/2}$

Using eqs. (11) we can calculate the derivatives of ϵ_x^2 and ϵ_y^2 (again assuming constant β):

$$\frac{d\epsilon_x^2}{dz} = 32K_{xy}(\Delta x^2\Delta x'y - \Delta xy\Delta xx')$$

$$\frac{d\epsilon_y^2}{dz} = 32K_{yx}(\Delta y^2\Delta xy' - \Delta xy\Delta yy') \qquad (13)$$

Since the rotated quadrupoles induce finite correlations between x and y the rms emittances are not conserved.

We may also define a quantity $l \equiv \Delta xy' - \Delta x'y$, which is proportional to the z component of the angular momentum. Again using eqs. (11) and some manipulations involving the definitions, and eqs. (4), (5) and (9), we find,

$$\frac{dl}{dz} = (K_{qyy} - K_{qxx})\Delta xy + K_{qxy}(\Delta x^2 - \Delta y^2) \qquad (14)$$

As can be seen from eq. 14, the angular momentum is not necessarily conserved when the quadrupoles are rotated. Physically, after a beam has passed through a quadrupole the beam will in general be elliptical. On passing through a quadrupole rotated relative to the first, the principal axis of the elliptical beam will not align with the quadrupole axes and a torque will be applied to the beam, causing a rotation of the beam. (Note also that eq. 14, does not depend on the self space-charge forces of the beam, as expected).

Because the focusing strength is a function of z, the effective external potential well within which the beam travels is z dependent, and so the transverse beam energy H is also not a constant in z. However, in the hard-edge model, within each quadrupole and drift section the focusing strength is assumed constant, and therefore the transverse energy is constant. We may use the result of ref. [2], adding the kinetic and potential energy terms to

obtain a total transverse energy. To obtain the potential energy of the beam in the external quadrupole field, we transform Δx_0^2 and Δy_0^2 to the lab frame. The result is

$$2H = \Delta x'^2 + \Delta y'^2 - \\ K_{qx0}\left((\Delta x^2 - \Delta y^2)\cos 2\theta + 2\Delta xy \sin 2\theta\right) - \\ K\ln\left((\Delta x_b^2)^{1/2} + (\Delta y_b^2)^{1/2}\right) \qquad (15)$$

Here Δx_b^2 and Δy_b^2 may be expressed in laboratory quantities using eqs. (7) and (8).

IV. "EMITTANCE-LIKE" CONSTANT OF THE MOTION

Although the emittance is not a constant with respect to z, a quantity which is related to the emittance is conserved. We define a generalized emittance ϵ_g by:

$$\epsilon_g^2 \equiv \frac{1}{2}\epsilon_x^2 + \frac{1}{2}\epsilon_y^2 + 16(\Delta xy\Delta x'y' - \Delta xy'\Delta x'y) \qquad (16)$$

It is readily shown using eqs. 11 and 13 that $\frac{\mathrm{d}\epsilon_g^2}{\mathrm{d}z} = 0$.

V. EXAMPLES OF RESULTS

A code was written to integrate eqs. (11). The results for ϵ_x and ϵ_g are plotted in figure 1 for a singly charged potassium beam ($A = 39$) with a current of 2 mA, an energy of 80 kV, initial emittances $\epsilon_x = \epsilon_y = 2.5 \times 10^{-5}$ m rad, $K_{qxo} = 30.9$ m^{-2}, and with $\langle\theta\rangle = 0.0234$ and $\Delta\theta^2 = 0.0156$. The occupancy of the quadrupoles was 0.33 and the half-lattice period was 0.36 m. The integration length was 40 half-lattice periods.
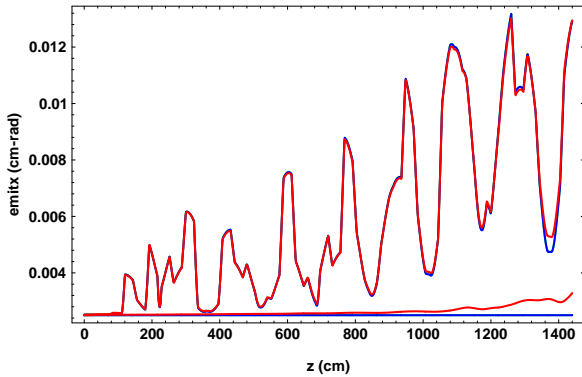

Figure 1. ϵ_x (oscillating) and ϵ_g (nearly constant) vs z for both integration of eq. 11, and particle-in-cell results.

Also shown in Figure 1 is a 2D particle-in-cell (PIC) simulation with the same parameters, for an initial distribution that is KV (ref.[3]), propagating through a pipe with circular cross section and 6 cm radius. The near identical overlap of the curves suggests that if the initial distribution is KV the assumption that the space charge field remains linear is at least a good approximation and possibly an exact result. The small increase in ϵ_g for large z is probably due to the non-linear image forces arising because of the finite pipe radius in the PIC simulations.

When these results are applied to the small recirculator of ref. [4], we find that with 2 mrad rms errors, there occurs only a 2%

increase in emittance for a beam which drifts (rather than accelerates) the nominal 15 laps. When the rotation errors are random over all 15 laps the emittance increases by about 50%. An accelerated beam will presumably show behavior somewhere in-between. A generalization of the theory presented here to include acceleration is in progress.

VI. DISCUSSION AND CONCLUSION

In inertial fusion applications, the ultimate goal is to focus the beam onto a small, 2-3 mm spot at the target. The final emittance is one of the important parameters needed to calculate the achievable final spot size (see e.g. ref.[5]). When quadrupole rotation errors are present, the beam will in general have a finite rotation angle and rotation rate, and will focus down to a more elliptical shape than in the absence of errors, reducing the power level that falls within a given spot radius. Analogous to the case of centroid displacements, it is conceivable that a system of intentionally rotated quadrupoles could compensate for the accumulated errors if the ten moments in eq. 11 are known.

In summary, we have used a formulation, in which the major assumption is that the space charge force can be calculated by assuming that the beam remains a uniform density ellipse with a shape that evolves in z. Under this assumption we have a derived a set of moment equations which generalizes the conventional envelope equations. We have found the misalignments cause the beam to acquire an overall angular momentum, and an increase in emittance measured relative to fixed laboratory axes. A generalized emittance has been constructed which is a conserved quantity (when the forces remain linear). Particle-in-cell results have shown agreement with the moment equations, and have suggested that the formulation may be exact if the initial distribution is KV. We have applied this method to estimate rotation alignment tolerances in the small recirculator of ref [4], and have suggested that this formulation will be useful when setting alignment tolerances and/or correction methods in an inertial fusion driver.

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