LONGITUDINAL COUPLING IMPEDANCE OF A HOLE IN AN INFINITE PLANE SCREEN*

Yong-Chul Chae
Argonne National Laboratory, 9700 So. Cass Ave., Argonne, Illinois 60439, U.S.A.

Abstract

An analytical formula for the longitudinal coupling impedance of a hole is developed using a variational method. We show that the coupling impedance can be expressed as a sum of functional series, whose argument is the dimensionless quantity kd alone, where k is the free-space wave number and d is the radius of the hole. When expanded in powers of kd, we recover the long wavelength result as a limiting case. The numerical evaluation reveals that the impedance can be well modeled by an RLC-resonator circuit. We also show the qualitatively good agreement between the theory and the MAFIA-T3 simulation for the geometry of a hole in a coupled waveguide with rectangular cross section.

I. PROBLEM STATEMENT

The geometry of our problem is shown in Fig. 1 where a charge is moving in the z-direction with velocity close to the speed of light. The distance between the plane screen and the beam path is b, and the origin of the coordinate is at the center of the hole with radius d. The local cylindrical coordinate system (ρ, θ, y) is also shown. We calculate the longitudinal coupling impedance for this geometry.

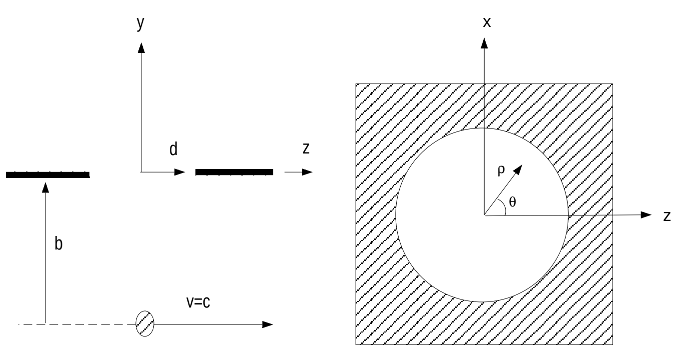

Figure 1. Infinite Flat Screen with a Hole.

II. LOW FREQUENCY SOLUTION

Denoting $\mathbf{E}_1$ and $\mathbf{H}_1$ as the fields without the hole and $\mathbf{E}_2$ and $\mathbf{H}_2$ as the fields with the hole, we can express the longitudinal coupling impedance as [1]

$$|I_0|^2 Z(k) = \int_{hole} (\mathbf{n} \times \mathbf{E}_2) \cdot \mathbf{H}_1^* dS, \tag{1}$$

where $\mathbf{n} \times \mathbf{E}_2 \equiv \mathbf{J}_m$ is the magnetic current induced in the hole, which is not known until we solve the problem.

Assuming a small hole, namely $kd = 2\pi d/\lambda \ll 1$, Bethe obtained the solution for the magnetic current in the hole as [2]

$$\mathbf{n} \times \mathbf{E} = \underbrace{-\frac{\rho}{\pi\sqrt{d^2-\rho^2}}\,\mathbf{e}_\rho \times \mathbf{E}_0}_{J_{m,E}} + \underbrace{j\frac{2kZ_0}{\pi}\sqrt{d^2-\rho^2}\,\mathbf{H}_0}_{J_{m,H}}, \tag{2}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are the field evaluated at the center of the hole in the absence of the hole, and $\mathbf{J}_{m,E}$ and $\mathbf{J}_{m,H}$ denote the magnetic current induced in the hole due to the incident electric and magnetic fields, respectively.

The magnetic field from the unit source current can be obtained using the image principle. In the plane of the hole, it becomes

$$H_x = -\frac{I_0}{\pi}\frac{b}{x^2+b^2}e^{-jkz}, \quad H_z = 0, \tag{3}$$

where the coordinate system defined in Fig. 1 is used.

Assuming a small hole in which the field strength is uniform but the phase is varying, we may rewrite the source field as

$$\mathbf{H}_1 = \mathbf{H}_0 - jkz\mathbf{H}_0 + \mathcal{O}(k^2), \text{ where } \mathbf{H}_0 = -\frac{I_0}{\pi b}\mathbf{e}_x. \tag{4}$$

Then, the longitudinal coupling impedance becomes

$$|I_0|^2 Z_H(k) = \int_{hole} \mathbf{J}_{m,H} \cdot \mathbf{H}_1^* dS = j\frac{4Z_0 d^3 H_0^2}{3}k, \tag{5}$$

$$|I_0|^2 Z_E(k) = \int_{hole} \mathbf{J}_{m,E} \cdot \mathbf{H}_1^* dS = -j\frac{2Z_0 d^3 H_0^2}{3}k, \tag{6}$$

$$|I_0|^2 Z(k) = Z_H(k) + Z_E(k) = j\frac{2Z_0 d^3 H_0^2}{3}k, \tag{7}$$

which results in $Z(k) = (2Z_0 d^3/3\pi^2 b^2)k$.

If we apply the above formula to a cylindrical beam pipe of radius b with a hole of radius d, the longitudinal coupling impedance becomes, with $H_0 = \frac{I_0}{2\pi b}$ in Eq. (7),

$$Z(k) = j\frac{Z_0 d^3}{6\pi^2 b^2}k, \tag{8}$$

which is exactly the same as the well-known results [1].

III. VARIATIONAL SOLUTION

A. Variational Formalism

We begin by defining an "impedance functional" which is stationary with respect to the unknown quantity (magnetic current density in the hole).

We define the impedance functional $\mathcal{Z}$ as

$$\mathcal{Z} = -\int \mathbf{J} \cdot (\mathbf{E}_2 - \mathbf{E}_1) dV. \tag{9}$$

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In the above definition, as we subtracted the contribution from the source field, the entire contribution is from the scattered field which satisfies the homogeneous Maxwell's equations. We note that if the electric field is real, the longitudinal impedance is the complex conjugate of the impedance functional, $Z(k) = \mathcal{Z}^*(k)^1$.

If the integrating surface is chosen to coincide with the plane of the screen where $\mathbf{E}_1$ satisfies the boundary condition $\mathbf{n} \times \mathbf{E}_1 = 0$, $\mathcal{Z}$ reduces to

$$\mathcal{Z} = \int \mathbf{H}_1 \cdot (\mathbf{n} \times \mathbf{E}_2) dS. \tag{10}$$

By using Rumsey's *reaction concept* [3], [4], we can derive the variational expression for $\mathcal{Z}$ as [5]

$$\mathcal{Z} = \frac{1}{4j\omega\epsilon} \frac{\left[\int_{S_a} \mathbf{H}^i \cdot (\mathbf{n} \times \mathbf{E}^a) dS\right]^2}{\int_{S_a}\int_{S_a} [\mathbf{n} \times \mathbf{E}^a(\mathbf{r})] \cdot \overline{\mathbf{G}}_0(\mathbf{r}|\mathbf{r}') \cdot [\mathbf{n} \times \mathbf{E}^a(\mathbf{r}')] dS dS'}, \tag{11}$$

where $\mathbf{H}^i$ is the incident magnetic field on the screen (previously denoted as $\mathbf{H}_1$), $\mathbf{E}^a$ is the assumed electric field in the hole, and $\overline{\mathbf{G}}_0(\mathbf{r}|\mathbf{r}')$ is the free-space dyadic Green's function.

The above formula is a homogeneous equation in the sense that the result does not depend on the amplitude of the assumed electric field $\mathbf{E}^a$. If a proper dyadic Green's function is used, this is a general expression for the impedance functional of an aperture in a conducting plane as long as the plane is the symmetry plane separating two regions, namely, an infinite plane or coupled waveguide structure. Details of the calculation depend on the shape of the aperture and the assumed tangential electric field in the aperture.

B. Results

In order to evaluate the variational expression represented by Eq. (11), we assume a trial function for $\mathbf{E}^a$ based on the Bethe's solution in Eq. (2):

$$\mathbf{n} \times \mathbf{E}^a = \mathbf{e}_\theta \sum_{n=1}^{n=\infty} b_n \rho^n (1 - \frac{\rho^2}{d^2})^{\frac{1}{2}-n} + \mathbf{e}_x \sum_{n=1}^{n=\infty} a_n (1 - \frac{\rho^2}{d^2})^{n-\frac{1}{2}}. \tag{12}$$

This field satisfies Meixner's “edge-field” condition [6].

The coefficients a_n and b_n are unknown quantity and dependent on the frequency. We used the method developed by Levine and Schwinger [7] to determine these coefficients, and the detailed results can be found in [5].

Once the a_n and b_n coefficients are determined, we can use them to calculate the longitudinal coupling impedance. It turns out that the coupling impedance is numerically equal to the impedance functional. We also found that the magnetic current from the electric and magnetic field does not couple in contribution to the coupling impedance. Thus we write the impedance as

$$Z^{(N+M)} = Z_E^{(N)} + Z_H^{(M)}, \tag{13}$$

where M or N denotes the order of approximation or the number of terms used for trial fields.

[1] We found that the coupling impedance does not have the stationary property in general.

It may be interesting to compare $Z_E^{(1)}(k)$ and $Z_H^{(1)}$ expanded in powers of kd. We find that

$$\begin{aligned} Z_H^{(1)} &= \frac{32 Z_0 d^2 H_0^2}{27\pi}(kd)^4 \left(1 + \frac{22}{25}(kd)^2 - \cdots\right) \\ &\quad + j\frac{4 Z_0 d^2 H_0^2}{3}(kd)\left(1 + \frac{2}{5}(kd)^2 - \cdots\right), \\ Z_E^{(1)} &= \frac{8 Z_0 d^2 H_0^2}{27\pi}(kd)^4 \left(1 - \frac{19}{25}(kd)^2 + \cdots\right) \\ &\quad - j\frac{2 Z_0 d^2 H_0^2}{3}(kd)\left(1 - \frac{2}{5}(kd)^2 + \cdots\right). \end{aligned}$$

In the low frequency range, it is found that $Z^{(2)} \simeq j(2Z_0 d^3 H_0^2/3)k$, which is the same as the low frequency result found in the previous section.

Since $H_0 \sim b^{-1}$, the above result shows that the impedance scales as $(d/b)^2$.

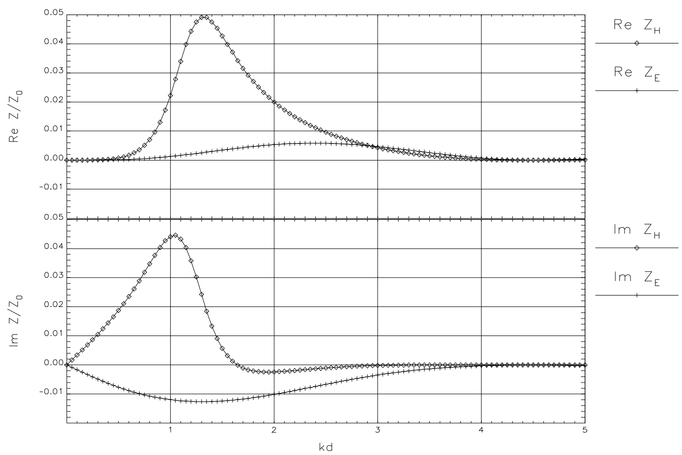

Figure 2. Comparison of Impedances due the Incident Magnetic Field, Z_H, and Electric Field, Z_E.

Numerical results of $Z_H^{(1)}$ and $Z_E^{(1)}$ are presented in Fig. 2. It shows that the impedance of magnetic type Z_H is mainly inductive ($Im\ Z_H > 0$), and the electric type Z_E exhibits capacitive behavior ($Im\ Z_E < 0$).

The results, using the three terms $Z^{(3)} = Z_E^{(1)} + Z_H^{(2)}$, are shown in Fig. 3, from which we find that the maximum value of $Re\ Z(k)$ is $Re\ Z(k)_{max} = 0.216 Z_0$. For all other d/b, it becomes $Re\ Z(k, d/b)_{max} = 0.216 Z_0 (d/b)^2$.

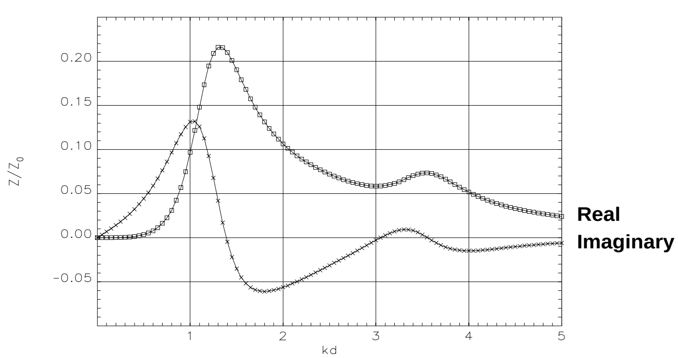

Figure 3. Variational Results Using Three-Term Electric Field. (The ratio d/b=1.0 is used.)

IV. BROADBAND RESONATOR MODEL

Since the impedance shown in Fig. 3 is similar to the impedance of a parallel RLC-resonator circuit, it would be useful if we described the impedance in terms of circuit parameters. The impedance of an RLC-resonator circuit is

$$Z(\omega) = \frac{R}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}, \tag{14}$$

where R is the shunt impedance, Q is the quality factor, and ω_r is the resonant frequency.

The resonant frequency and the quality factor can be read from Fig. 3. For the Q value, we used the definition $Q = \omega_r/2\Delta\omega$, where $|Z(\omega)|$ at the frequency $\omega = \omega_r + \triangle\omega$ is 0.707 of its maximum value. The shunt impedance can be determined in two ways. We can either use the impedance in the low frequency limit, $Z(k) = j(2Z_0 d^3 H_0^2/3)k$, or the impedance at resonance, $Z(k) = 0.216 Z_0 (d/b)^2$. Denoting these two models as BBR-1 and BBR-2, respectively, the circuit parameters are shown in Table I.

Table I
Circuit Parameter Based on BBR Model

Model	ω_r	Q	R
BBR-1	1.35(c/d)	1.8	0.164 $Z_0(d/b)^2$
BBR-2	1.35(c/d)	1.8	0.216 $Z_0(d/b)^2$

We compared the impedances from the two models with the variational result, which is shown in Fig. 4. Note that $(Z(k)/Z_0)/kd$ is plotted, which is the useful quantity in the instability calculation.

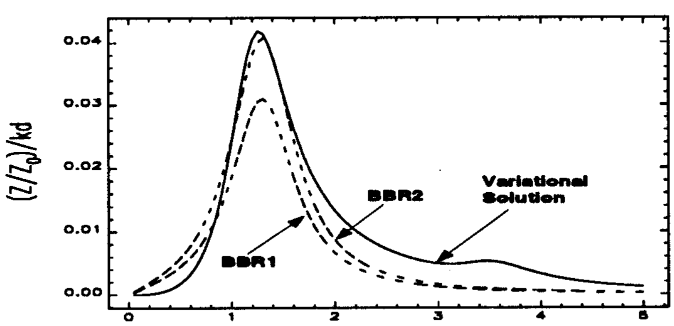

Figure 4. Comparison of Impedances from Variational Solution and Broadband Resonator Model.

V. APPLICATION TO ACCELERATOR CHAMBER

We also applied the above results to the accelerator chamber. As a model geometry we considered the rectangular waveguides coupled by the hole in the common wall. In the analysis, we used the image charges in order to remove the waveguide wall. By doing so we could investigate the contributions from the real charge and the image charges to the impedance separately. We found that the image-charge contribution is small, as long as d/b is small [5].

We compared the variational results with a MAFIA-T3 [8] simulation. The geometry used in the MAFIA-T3 simulation has a 2 cm-by-1 cm rectangular waveguide with a hole of varying radius on the 1-mm-thick common wall.

The results for the hole with a radius of 1 mm corresponding to d/b=0.2 are shown in Fig. 5. The agreement between the two results is qualitatively good. From the range of frequency we can conclude that the appropriate length scale is the size of the hole and not the size of the waveguide. Thus the scaling we found in the previous section also applies to the waveguide geometry.

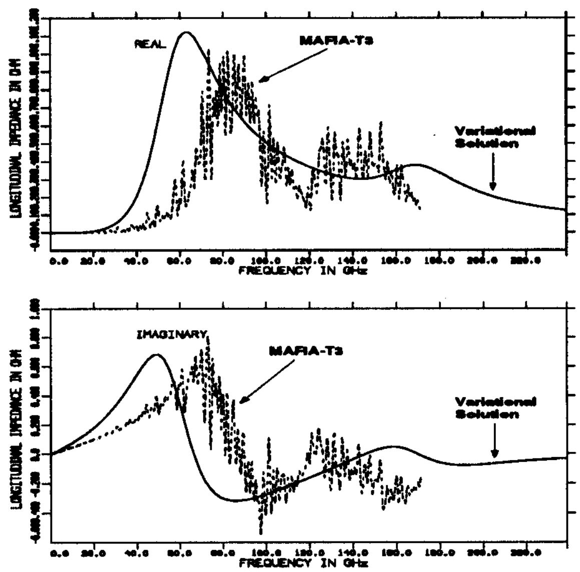

Figure 5. Coupling Impedance of the Hole with a Radius of 1 mm in the Coupled Waveguide.

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