Longitudinal Wakefield for Synchrotron Radiation

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ABSTRACT

It is of interest to determine how short a bunch can be maintained in a storage ring. Some aspects of this problem depend on specific details of the storage ring design. Instability thresholds depend on the strength of the wakefields. In this paper we determine the longitudinal wakefield due to curvature of the electron orbit, both in free space and between parallel plates with infinite conductivity. Our study leads to the conclusion that as long as the vertical aperture of the vacuum chamber is not too small, the wakefield due to image charges does not significantly vary along a short bunch. Therefore, we suggest that in the determination of the equilibrium bunch length of very short bunches, the effect of the conducting plates can be neglected. The starting point for bunch lengthening calculations should be the free space wakefield.

I. INTRODUCTION

In his pioneering book, published in 1912, Schott [1] calculated many of the properties of the radiation due to a relativistic point charge moving with uniform speed on a circular orbit. In particular, Schott determined the electromagnetic field at all points on the circular orbit, yielding what we now refer to as the wakefield of the point charge. Here, it is our goal to extend Schott's work in several directions, considering a highly relativistic particle whose velocity is close to the speed of light c. In this case the longitudinal wakefield simplifies and is large in front of the point charge and reduced by a factor of $1/\gamma^4$ behind. Working in the time domain, we obtain an analytic expression for the longitudinal wake in front of the charge. Next, we use the method of image charges to derive the longitudinal wake due to a point charge moving on a circular orbit lying between two parallel plates of infinite conductivity.

II. WAKEFIELD IN FREE SPACE

The retarded Lienard-Wiechert potentials and fields for a point particle are derivable from the time dependent Green function [2],

$$G(\vec{x},t;\vec{x}',t') = \delta\left(t' - t + |\vec{x}-\vec{x}'|/c\right) \Big/ |\vec{x}-\vec{x}'| \,. \tag{1}$$

In Lorentz gauge, the scalar and vector potentials are,

$$\Phi(\vec{x}',t) = \int d^3x'dt'\rho(\vec{x}',t)G(\vec{x},t;\vec{x}',t') \,, \tag{2a}$$

$$\vec{A}(\vec{x}',t) = \int d^3x'dt'\vec{j}(\vec{x}',t)G(\vec{x},t;\vec{x}',t') \,. \tag{2b}$$

For a point particle following a trajectory $\vec{r}_0(t)$ the charge and current densities are $\rho(\vec{x}',t) = e\delta\left(\vec{x}' - \vec{r}_0(t)\right)$ and $\vec{j}(\vec{x}',t) = e\vec{v}_0\delta\left(\vec{x}' - \vec{r}_0(t)\right)$, where $\vec{v}_0 = c\vec{\beta}_0(t) = d\vec{r}_0/dt$ is the instantaneous velocity.

Choosing $\vec{r}_0(t)$ to be the trajectory of an electron moving with uniform velocity on a circle of radius ρ, Schott determined the tangential electric field at the observation point $\vec{x}$ using

$$E_\phi(\vec{x},t) = -\frac{1}{\rho}\frac{\partial}{\partial\phi}\Phi(\vec{x},t) - \frac{1}{c}\frac{\partial}{\partial t}A_\phi(\vec{x},t) \,. \tag{3}$$

Starting from Schott's result we consider its simplification in the limit as $\beta \to 1$. The tangential electric field on the circular orbit is written as the sum of the singular Coulomb term and a non-singular term, $\tilde{E}_\phi$ due to synchrotron radiation:

$$E_\phi = \frac{e}{4\rho^2\gamma^2}\frac{\cos\xi}{\sin^2\xi} + \tilde{E}_\phi \,. \tag{4}$$

Here, γ is the electron energy measured in units of its rest mass and $2\xi = s/\rho$ is the angle between the electron and the observation point. Directly in front of the electron $\xi = 0$ and directly behind $\xi = \pi$. Introducing the scaled angle, $\mu \equiv 3\gamma^3\xi$, the non-singular term is given by,

$$\tilde{E}_\phi(\mu) = -\frac{2U_0}{2\pi\rho e}\cdot\frac{d\Psi(\mu)}{d\mu} = \tag{5}$$

$$-\frac{4e\gamma^4}{3\rho^2}\begin{cases} 0, & \mu < 0 \\ \frac{1}{2}, & \mu = 0 \\ \frac{d}{d\mu}\left[\frac{9}{4}\cdot\frac{\cosh\left[\frac{5}{3}\sinh^{-1}\mu\right] - \cosh\left[\sinh^{-1}\mu\right]}{\sinh\left[2\sinh^{-1}\mu\right]}\right], & \mu > 0 \end{cases}$$

where $U_0 = 4\pi e^2\gamma^4/3\rho$ is the energy loss per turn for a single electron. Since $\Psi(\mu)$ vanishes at μ = 0 and μ = ∞, the function $d\Psi(\mu)/d\mu$ has the additional property, $\int_0^\infty d\Psi(\mu) = 0$; a plot of $\Psi(\mu)$ & $d\Psi(\mu)/d\mu$ is given in Figure 1. Useful asymptotic expansions for small and large μ > 0 are [3-5],

$$\frac{d\Psi(\mu)}{d\mu} \equiv W(\mu) = \begin{cases} 1 - \frac{14}{9}\mu^2 + \ldots, & \mu << 1 \\ -\frac{3}{4\cdot 2^{1/3}}\frac{1}{\mu^{4/3}} + \ldots, & \mu >> 1 \end{cases} . \tag{6}$$

The wake function is discontinuous at the position of the point charge. The choice, $d\Psi(0)/d\mu = 1/2$, represents an average of the electric field immediately in front and behind the charge, and results in the power loss being correctly given by $ec\tilde{E}_\phi(0)$. The total power radiated by an individual electron is $P = I_0 \cdot U_0 = 2ce^2\gamma^4/3\rho^2$.

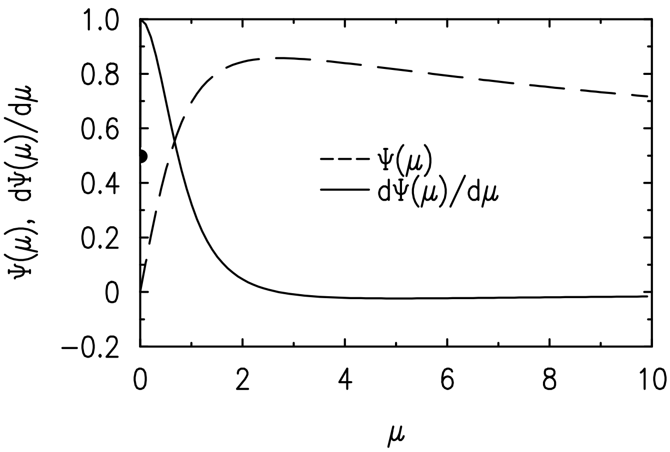

Figure 1: Free space potential function Ψ and wakefield function $d\Psi/d\mu$.

The wake potential due to synchrotron radiation that is generated by a beam of electrons is then given as,

$$\tilde{V}(s) = \int_{-\infty}^{\infty} \tilde{E}_\phi(s - s') \cdot I(s')ds' . \quad (7)$$

For a Gaussian electron beam the curvature wake potential, which is plotted in Figure 2, scales as [5],

$$\tilde{V}(s) \propto \frac{1}{\sigma^{4/3}} \exp\left[-\frac{s^2}{4\sigma^2}\right] D_{1/3}(-s/\sigma) , \quad (8)$$

where $D_{1/3}(x)$ is a parabolic cylinder function.

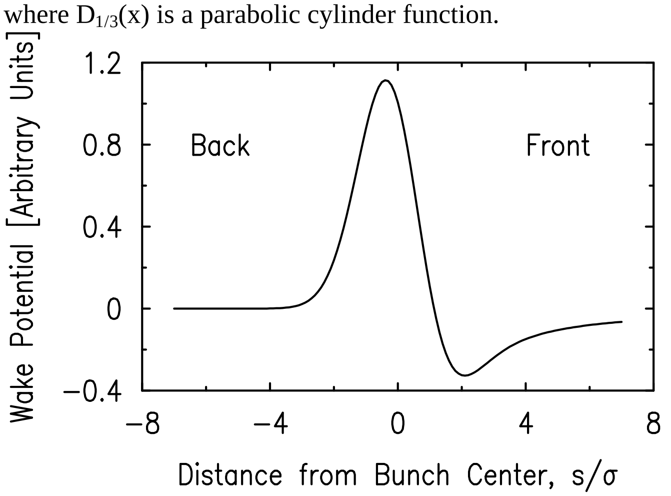

Figure 2: Wake Potential for a Gaussian bunch.

III. WAKEFIELD FOR PARALLEL PLATES

Let us now consider an electron moving with uniform speed on a circular orbit in the horizontal midplane, between two parallel plates of infinite conductivity located at $z = \pm h$. Using the method of image charges we easily see that the free space Green function of equation (1) should be replaced by the parallel plate Green function,

$$G_{PP}(\vec{x}, t; \vec{x}', t') = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \delta\left(t' - t + \frac{|\vec{x} - \vec{x}' - 2nh\hat{z}|}{c}\right)}{|\vec{x} - \vec{x}' - 2nh\hat{z}|} . \quad (9)$$

Here the observation point $\vec{x}$ and the source point $\vec{x}'$ lie in the midplane and $\hat{z}$ is a unit vector in the vertical direction perpendicular to the plates.

Using the Green function of equation (9) together with equation (2-3), we proceed in a manner analogous to that of the free space calculation. Introducing the parameter $\Delta \equiv h/\rho$, the tangential electric field on the orbit is found to be given by ($\Delta << 1, \gamma^2\Delta >> 1$):

$$E_\phi \cong \frac{e}{4\rho^2\gamma^2}\left[\frac{\cos\xi}{\sin^2\xi} + \frac{1}{\Delta^3} G_1(\xi/\Delta^{3/2})\right] - \frac{4}{3}\frac{e\gamma^4}{\rho^2}\left[W(3\gamma^3\xi) - \frac{3}{8}\frac{1}{\Delta^2\gamma^4} G_2(\xi/\Delta^{3/2})\right] \quad (10)$$

where $W(\mu) \equiv d\Psi(\mu)/d\mu$, and the scaling functions G_1 and G_2 are plotted in Figure 3.

In free space the radiation travels along a chord to another point on the circle, thereby always arriving before (in front) of the exciting charge. With the plates in place the radiation can bounce off the plates once or numerous times and arrive behind the exciting charge resulting in a trailing wakefield, which we have described using image charges. For $\gamma^2\Delta >> 1$, the G_1 term is smaller than the G_2 term and for small ξ they are both small compared to the free space wakefield, $W(3\gamma^3\xi)$. The main effect of G_2 is to cancel the tail of $W(3\gamma^3\xi)$ at large distances (times), which is the time domain analog of the suppression of low frequencies in the impedance due to the parallel plates. The G_1 term cancels the long distance tail of the Coulomb term. When $\gamma^2\Delta \approx 1$, the G_1 , G_2 and W term become comparable in magnitude, and the situation is more complicated.

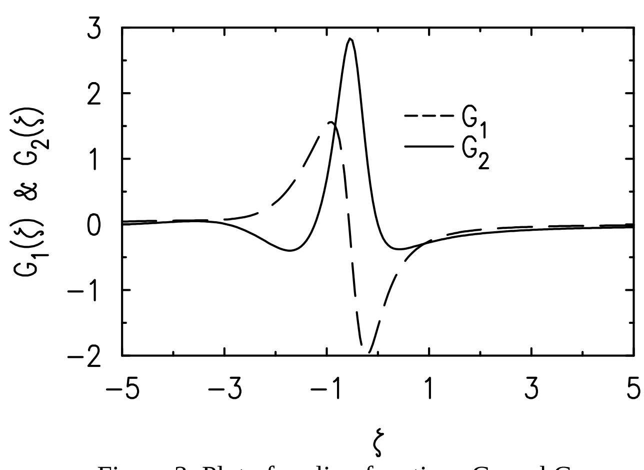

Figure 3: Plot of scaling functions G_1 and G_2.

The expression for E_ϕ in equation (10) is given in the time domain. Results in the frequency domain were obtained in references [6,7]. We believe that the time domain results are more directly applicable to the study of the bunch lengthening instability.

We have also calculated the tangential electric field at the position of the charge itself for $\Delta << 1, \gamma^2\Delta >> 1$, we have found the radiated power loss $P = ecE_\phi(0)$ to be given by,

$$P \cong \frac{2}{3}\frac{ce^2\gamma^4}{\rho}\left[1+\frac{3\sqrt{3}K_1}{8}\frac{1}{\gamma^6\Delta^3}+\frac{3\sqrt{3}K_2}{10}\frac{1}{\gamma^4\Delta}\right], \qquad (11)$$

where $K_1 \cong \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}$ and $K_2 \cong \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln 2$.

In equation (11) we have averaged the singular free space contribution to the electric field immediately in front and behind the point charge. The second term, proportional to K_1, is in agreement with the G_1 term in equation (10). The third term cannot be obtained from equation (10) because $G_2(0) = 0$. It was necessary to go to higher order in Δ. The correction due to the conducting plates is small in magnitude and has the sign indicating that the electron radiates more energy between the conducting plates than it would in free space. This is the case despite the fact that the plates cut off the lowest frequencies.

IV. LONGITUDINAL COUPLING IMPEDANCE

The longitudinal coupling impedance due to synchrotron radiation is related to the Fourier transform of the non-singular function wake, $Z_n \equiv 2\pi\rho\cdot\tilde{E}_n / I_n$, where $\tilde{E}_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-in\theta}\tilde{E}_\phi(\theta)$ and for a single circulating charge, $I_n = ec\beta / 2\pi\rho$. In free space the impedance associated with this wake is computed to be in MKS units [8],

$$Z_n = Z_0\pi\beta n\begin{Bmatrix} [J'_{2n}(2n\beta) - iE'_{2n}(2n\beta)] \\ -\frac{n}{\beta^2\gamma^2}\int_0^{\beta}[J_{2n}(2nx) - iE_{2n}(2nx)]dx \\ -\frac{i}{\pi\gamma^2\beta^2}\ln\gamma \end{Bmatrix} \qquad (12)$$

where the Weber function, $E_n(z)$, is defined as $E_n(z) \equiv \frac{1}{\pi}\int_0^{\pi}\sin(nt - z\sin t)dt$.

The real part of the impedance is proportional to the synchrotron radiation power emitted at a frequency $n\omega_0$ and integrated over all vertical angles for a single electron,

$$\mathrm{Re}\, Z_n = Z_0\pi\beta\left[nJ'_{2n}(2n\beta) - \frac{n^2}{\beta^2\gamma^2}\int_0^{\beta}J_{2n}(2nx)dx\right] = \frac{2P_n}{I_n^2} \qquad (13)$$

For $1<<n<<\gamma^3$, only the first two terms in the expression for the impedance are significant which implies the far field radiation term is dominant. Using the expressions for J_n and E_n when the order and argument are nearly equal one can recover the well known result of Faltens & Laslett [9],

$$Z_n \approx Z_0\cdot\frac{\Gamma(2/3)}{3^{1/3}}\cdot\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)\cdot n^{1/3}, \qquad (14)$$

except the sign of the reactive part is opposite to that given by them since the wake is in front of the exciting charge and not behind.

For $n >> \gamma^3$, the first four terms in the expression for the impedance are now significant which implies that at very high frequencies, characteristic of very short distances, the near field also contributes to the impedance.

V. ACKNOWLEDGMENTS

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