Mode-Coupling Instability and Bunch Lengthening in Proton Machines

K.Y. Ng

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510*

Abstract

In proton machines, potential-well distortion leads to small amount of bunch lengthening with minimal head-tail asymmetry. Longitudinal mode-mixing instability occurs at higher azimuthal modes. When the driving resonance is of broad-band, the threshold corresponds to Keil-Schnell criterion for microwave instability. [1] When the driving resonance is narrower than the bunch spectrum, the threshold corresponds to a similar criterion derived before. [2]

I. Introduction

Proton bunches are very much different from electron bunches. First, electron bunches have a length roughly equal to the radius of the beam pipe, whereas proton bunches are usually very much longer. Second, the momentum spread of the electron bunches is determined by the heavy synchrotron radiation. Protons do not radiate and behave quite differently in the longitudinal phase space, with the bunch area conserved instead. These differences lead to different results in potential-well distortion and mode mixing, which we will discuss briefly below. The details are given in a separate paper. [3]

II. Potential-Well Distortion

As an example, the bunches in the Fermilab Main Ring have a typical full length of $\sim$ 60 cm or $\tau_L \approx 2$ ns. The spectrum has a half width of $\sim$ 0.5 GHz. Therefore, the static bunch profile is hardly affected by the resistive part of the broad-band impedance which is centered at 1.5 $\sim$ 2 GHz. As a result, the inductive part of the broad-band will only lead to a symmetric broadening (shortening) of the bunch above (below) transition. Numerically solving the Haïssinski equation [4] confirms this conjecture. Strictly speaking, the Haïssinski equation does not apply to proton bunches where the bunch area is conserved and the momentum spread is not a fixed Gaussian.

Since the driving impedance is inductive, the wake potential is the derivative of the δ-function. For a parabolic bunch, the wake force will be linear and can be superimposed onto the linearized rf force easily. We use for the distribution in longitudinal z-δ phase space, [5]

$$\psi(z,\delta)=\frac{3\eta cN}{2\pi\omega_{s0}\hat{z}_0^3}\sqrt{\hat{z}_0^2-\frac{1}{\kappa}\left(\frac{\eta c}{\omega_{s0}}\delta\right)^2-\kappa z^2}\;, \tag{2.1}$$

where η is the phase-slip parameter, c the velocity of light, $\omega_{s0}/2\pi$ the unperturbed synchrotron frequency, and N the number of particles in the bunch. The original half length of the bunch $\hat{z}_0$ has been lengthened to $\hat{z}_0/\sqrt{\kappa}$, whereas the momentum spread δ is shortened by $\sqrt{\kappa}$, so that the bunch area remains the same. The Hamiltonian is modified to

$$H=\frac{\eta^2c^2}{2\omega_{s0}}\delta^2+\frac{\omega_{s0}}{2}(1-D\kappa^{3/2})z^2\;, \tag{2.2}$$

where

$$D=\frac{3e^2N\eta c^3}{4\pi\omega_{s0}^2E\hat{z}_0^3}\left.\frac{Z}{n}\right|_{\rm ind}\;. \tag{2.3}$$

The incoherent synchrotron angular frequency is therefore $\omega_s=\omega_{s0}(1-D\kappa^{3/2})^{1/2}$. Since the distribution $\phi(z,\delta)$ must be a function of the Hamiltonian, we obtain the constraint $\kappa^2=1-D\kappa^{3/2}$. Take a Fermilab Main Ring bunch with $N=2.5\times10^{10}$ at $E=150$ GeV, bunch area 0.15 eV-sec, and synchrotron tune $\nu_{s0}=0.0030$. The accelerator ring has a revolution frequency $f_0=47.7$ kHz, a phase-slip parameter $\eta=0.0028$, and the inductive impedance is believed to be $Z/n|_{\rm ind}\approx 10$ Ohms. Then $D=0.204$, indicating that the bunch has been lengthened by $\kappa^{-1/2}=1.05$ and the momentum spread flattened by 5%. This implies that we cannot infer the the momentum spread by naively measuring the bunch length and the synchrotron frequency through the relation $\hat{\delta}=\omega_{s0}\tau_L/\eta$, because the answer will be 10% too large, giving a wrong idea about the amount of Landau damping. Instead, the momentum spread should be measured from Schottky signals or inferred through dispersion from the measurement of the transverse profile of the bunch using a flying wire.

III. Mode-Mixing

The shifts of the synchrotron side-bands can be derived using Vlasov equation. Here, we follow the Sacherer's approach. [6] The coherent side-band synchrotron angular frequencies ω can be obtained from the determinant

$$|(\omega-m\omega_s)\delta_{mm'}-M_{mm'}|=0\;. \tag{3.1}$$

The longitudinal impedance $Z(n)$ in the matrix elements $M_{mm'}=\bar{\epsilon}\omega_sA_{mm'}$ is responsible for the coupling of the azimuthal modes, with

$$A_{mm'}=\frac{m}{m+1}\frac{\sum_n[jn_r\hat{Z}(n)/n]h_{mm'}(n)}{\sum_n h_{mm}(n)}\;, \tag{3.2}$$

where $h_{mm'}(n)=\tilde{\lambda}_m^*(n)\tilde{\lambda}_{m'}(n)$ are the overlap of the spectral function $\tilde{\lambda}_m(n)$ of the bunch obtained by solving the matrix $M_{mm'}$. In the above, $\bar{\epsilon}=\epsilon(\omega_{s0}/\omega_s)^2$, $\epsilon=I_b(R_s/n_r)/(3B_0^3hV\cos\phi_s)$, I_b the average bunch current, $\hat{Z}$ the resonant impedance centered at $f_r=n_rf_0$ and normalized to the shunt impedance R_s, V unperturbed rf voltage, ϕ_s the synchronized phase, h the rf harmonic, $B_0=\tau_Lf_0$ the bunching factor, and τ_L the full bunch length in sec.

Potential-well distortion has been neglected in the formulation, because the effect is small for proton machines. We will use a prescribed set of $\tilde{\lambda}_m(p)$ instead of the eigenvectors. Although

* Operated by the Universities Research Association under contracts with the U.S. Department of Energy.

self-consistency will be lost, we do think that the essence of the results will not be affected. We use Sacherer's sinusoidal bunch profiles. The spectral functions are

$$\tilde{\lambda}_m(p) = (-j)^m \frac{m+1}{2\pi} \frac{\cos \pi x/2}{x^2 - (m+1)^2} , \tag{3.3}$$

when m is even and with cosine replaced by sine when m is odd. A dimensionless frequency parameter $x = 2nf_0\tau_L$ has been introduced, so that, with the exception of $m = 0$, the spectrum for the mth mode peaks at $x \approx m+1$. Continuing with the example of the Fermilab Main Ring which has a broad band impedance centered at $x_r = 7.5$ or $f_r \sim 1.88$ GHz and $Q \approx 1$, we find the colliding of modes 6 and 7 in Fig. 1, and the bunch becomes unstable at $\epsilon = 0.94$. This is expected, because the symmetries of Eqs. (3.2) and (3.3) show that $\mathrm{R}eZ$ is responsible for the coupling between two adjacent modes. Note that the ordinate of Fig. 1 is normalized with respect to the unperturbed synchrotron frequency ω_{s0}, and an adjustment for the incoherent tune shift has been made.

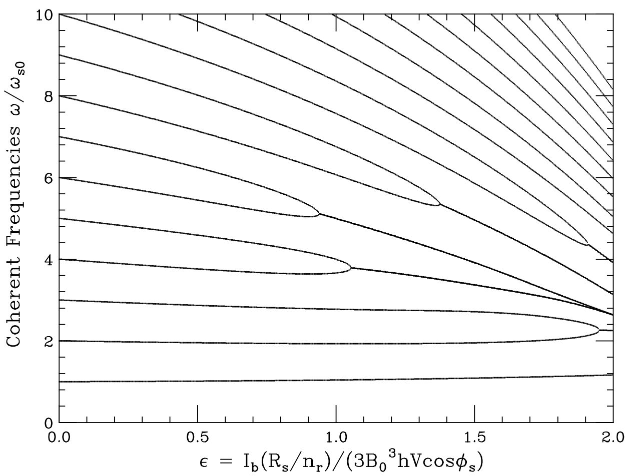

Fig. 2. Coupling of modes $m = 6$ and 7 in the presence of a resonance at $x_r = 7.5$ and $Q = 1$.

We vary Q and compute the threshold ϵ_{th} in each case. The result is plotted in Fig. 2 versus $z = \Delta f_r \tau_L = x_r/4Q$, where $\Delta f_r = f_r/2Q$ is the HWHM of the resonance. Also plotted are threshold curves at different resonant frequencies x_r. Note that all the curves approach a minimum threshold of $\epsilon_{th} \approx 0.92$ at $z \approx 0.6$. The latter has the physical meaning of the resonance peak just wide enough to cover only two coupling modes. A smaller z implies that the resonance peak is too narrow and interacts with only parts of the two mode spectra, thus giving a higher instability threshold. A larger z means that the resonance will cover more than two mode spectra. For $x_r = 7.5$ say, modes 6 and 7 will then be pulled and pushed also by the other modes, so the threshold for their collision is expected to be higher also. However, Eq. (2.2) reveals that the coupling comes in not through $\mathrm{R}eZ(n)$ but through $\mathrm{R}eZ(n)/n$, whose peak value becomes larger and the peak frequency smaller when Q is small, as illustrated in Fig. 3. As a result, the lower modes start to collide first (Fig. 4). Thus the threshold for large z remains small, which is very much different from what Sacherer stated in his paper.

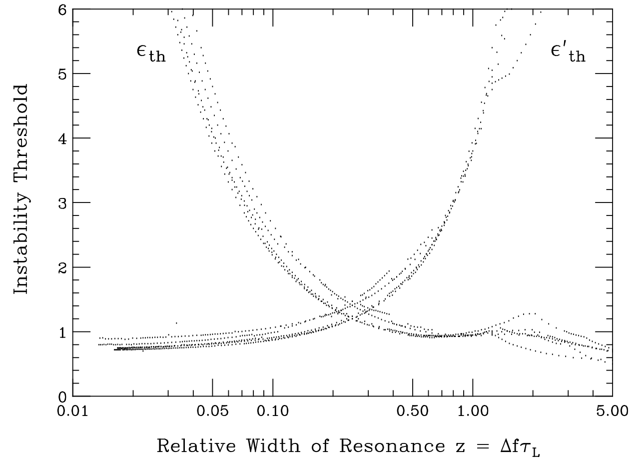

Fig. 2. Instability thresholds ϵ_{th} and ϵ'_{th} for various widths of the resonance impedance located at $x_r = 3.5$ to 11.5.

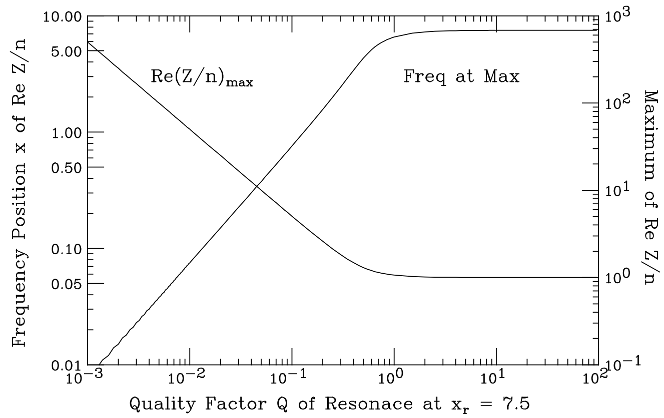

Fig. 3. Enhancement of $(\mathrm{R}eZ/n)_{\max}$ (normalized to R_s) and its frequency position x as the quality factor Q of the resonance centered at $x_r = 7.5$ decreases.

IV. Microwave Instability Driven by Broad Resonances

Microwave instability can occur when the resonance is much wider than the bunch spectrum. When this happens, many coherent modes are excited. Therefore the threshold at the $z \gg 1$ end, $\epsilon_{th} \approx 0.75$, is the threshold of microwave instability. This threshold condition can be easily rewritten in terms of the energy spread $(\Delta E)_{\rm FWHM} = \frac{3}{2}(\Delta E)_{\rm full}$ and peak bunch current $I_p = \pi I_b/2\tau_L f_0$ of the sinusoidal profile as

$$\frac{R_s}{n_r} = \frac{3}{4}\epsilon_{th} \frac{\eta(E/e)}{I_p} \left(\frac{\Delta E}{E}\right)^2_{\rm FWHM} . \tag{4.1}$$

This is the familiar Boussard-modified Keil-Schnell criterion [1] of microwave instability driven by a broad resonance. The form factor for this type of bunch shape should be slightly bigger than unity, which is very close to $\frac{27}{16}\epsilon_{th} = 1.3$ obtained here. The equivalence of mode-coupling and microwave instability had been pointed out by Sacherer [6] and Laclare. [7]

When $z \approx 0.6$, $\mathrm{R}eZ$ is just wide enough to cover two adjacent azimuthal modes m and $m' = m+1$, and the excitation is one with

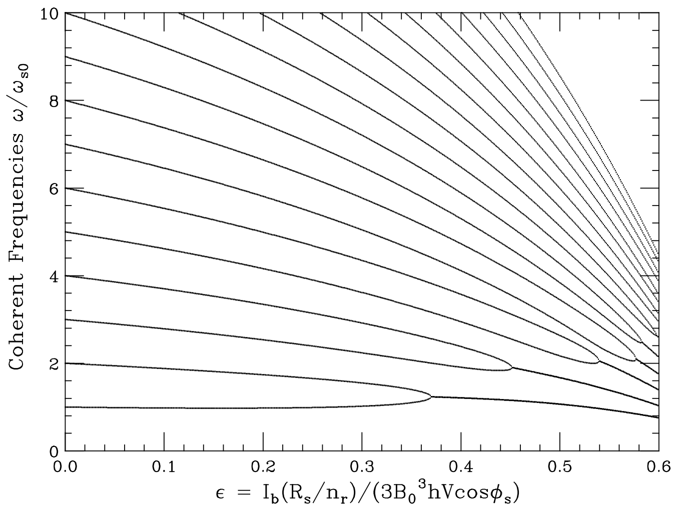

Fig. 4. Mode coupling starts at the lowest modes when the driving resonance is much wider than the bunch spectrum. Here $x_r = 7.5$, $Q = 0.2$, $\tau_L = 2$ ns, or $z = 37.5$.

$x_r = \frac{1}{2}(m+3)$ nodes along the bunch. The coupling matrix can be truncated to include only these two modes. From Eq. (3.1), the eigen modes are

$$\frac{\omega}{\omega_s} = \frac{1}{2}\left[(\nu_m + \nu_{m'}) \pm \sqrt{(\nu_{m'} - \nu_m)^2 - 4\bar{\epsilon}^2 A_{mm'}^2}\right] , \qquad (4.2)$$

where $\nu_k = k + \bar{\epsilon} A_{kk}$, $k = m$ or m'. The threshold of instability ϵ_{th} is therefore given by

$$|\bar{\epsilon}_{th} A_{mm'}| = \frac{1}{2}|\bar{\epsilon}_{th}(A_{m'm'} - A_{mm}) - 1| \,. \qquad (4.3)$$

The matrix elements A_{mm}, $A_{m'm'}$, and $A_{mm'}$ have been computed numerically for any two adjacent m, m', with the resonance peak centered at $x_r = \frac{1}{2}(m+3)$. The result is actually very close to $\epsilon_{th} = 0.92$ and depends on m very weakly. It can also be estimated easily. Since $A_{m'm'} \approx A_{mm}$, we have $|\epsilon_{th} A_{mm'}| \approx \frac{1}{2}$. If we further approximate the resonance and adjacent spectra by rectangular curves, we get $|A_{mm'}| \approx 0.5$.

V. Microwave Instability Driven by Narrow Resonances

When the resonance is much narrower than the width of the bunch spectrum, we have $z \ll 1$. Then, the summation over frequency in Eq. (3.2) can be approximated by

$$\sum_n \frac{x_r \hat{Z}(n)}{n} h_{mm'}(n) \approx \frac{\pi x_r}{Q} \tilde{\lambda}_m^* \tilde{\lambda}_{m'}|_{x=x_r} \,. \qquad (5.1)$$

For this, we need a new dimensionless current parameter $\epsilon' = 2I_b(R_s/Q)/(3B_0^2 hV\cos\phi_s)$. This new threshold ϵ'_{th} is now plotted versus z in Fig. 2. For small z, we obtain $\epsilon'_{th} \approx 0.75$ which is almost independent of x_r. Again, this threshold can be computed numerically using the truncated 2×2 coupling matrix, or estimated by approximating the spectral functions by rectangular curves. When it is cast into the form

$$\frac{R_s}{Q} = \frac{27}{16\pi}\epsilon'_{th}\frac{\eta(E/e)}{I_b}\left(\frac{\Delta E}{E}\right)^2_{\rm FWHM} , \qquad (5.2)$$

it is just the criterion of microwave instability driven by an impedance resonance that is narrower than the bunch spectrum. [2] The form factor is 0.41, which agrees very well with $\frac{27}{16\pi}\epsilon'_{th} \approx 0.40$. This may be a more appropriate microwave instability threshold for electron machines, since electron bunches are short.

VI. Going Below Transition

Figure 1 shows that the coherent frequencies tend to cluster together when the current ϵ increases. This is because we are above transition, $\cos\phi_s < 0$. Looking into the diagonal elements of Eq. (3.2), modes with $m < x_r - 1$ ($> x_r - 1$) sample the inductive (capacitive) part of the impedance and are shifted upward (downward). Below transition, the shifts will be in the opposite direction; i.e., diverging outward with increasing ϵ. In other words, the mode-mixing threshold ϵ_{th} will be increased, or the bunch becomes more stable. We tried to reverse the sign of $\cos\phi_s$ in the example of Fig. 1 and found ϵ_{th} increases from 0.94 to 1.88. Therefore, a bunch in a machine with a negative momentum-compaction factor [8] will be more stable. This idea had been pointed out by Fang et al [9] in shortening electron bunches.

References

[1] D. Boussard, CERN/LAB II/RF/75-2 (1975); E. Keil and W. Schnell, CERN/ISR/TH/RF/69-48 (1969)

[2] K.Y. Ng, Proc. 1986 Summer Study on Phys. of SSC, ed. R. Donalson and J. Marx, 1986, p.590.

[3] K.Y. Ng Fermilab Internal Report FN-630, 1995.

[4] J. Haïssinski, Nuovo Cimento **B18**, 72 (1973).

[5] A.W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, Wiley, p.284.

[6] F.J. Sacherer, IEEE Trans. Nucl. Sc. **24**, No.3, 1393 (1977).

[7] J.L. Laclare, CERN Accel. School, Queen's College, Oxford, England, 1985, p.264.

[8] S.Y. Lee, K.Y. Ng, and D. Trbojevic, Phys. Rev. **E48**, 3040 (1993).

[9] S.X. Fang, K. Oide, Y. Yokoya, B. Chen, and J.Q. Wang, KEK Preprint 94-190, 1995.