PRACTICAL CRITERION OF TRANSVERSE COUPLED–BUNCH HEAD–TAIL STABILITY

S. Ivanov and M. Pozdeev, IHEP, Protvino, Moscow Region, 142284, Russia

Abstract

Analytical outcome of the paper is a few formulae to simplify practical threshold calculations of transverse coupled-bunch head-tail instability caused by narrow-band impedances in a proton synchrotron, which provide a useful quantitative view on how to keep the instability under control with chromaticity and cubic-nonlinearity correctors of the magnetic field. The formulae include: (i) the envelopes of head-tail mode formfactors expressed via a pair of averages over a bunch longitudinal distribution, and (ii) expressions of the effective betatron tune spread introduced by partial spreads in 2-D function $\omega_y(\mathcal{J}_x, \mathcal{J}_z)$ of transverse action variables in x, z-directions, $y = x, z$. The tolerable values of transverse coupling impedances at parasitic higher-order E_{1np}-modes of the UNK accelerating cavities are estimated as an example of application.

I. INTRODUCTION

Let x, z be horizontal and vertical displacements from the orbit, $\vartheta = \Theta - \omega_0 t$ be azimuth in a co-rotating frame, where Θ is azimuth around the ring in the laboratory frame, ω_0 is the angular velocity of a reference particle, t is time. For definiteness, only x-oscillations are studied, the results being applicable to z-direction by $x \to z$ and v.v. Introduce a 6-D phase-space of variables $(y, y' \equiv dy/dt)$ with $y = \vartheta, x, z$. Let the unperturbed motion be integrable, x, z-oscillations being assumed uncoupled by the optics and treated in a 'smooth' approximation. Pass from (y, y') to angle-action variables $(\psi_y, \mathcal{J}_y)$, $y = \vartheta, x, z$ with $\omega_y(\mathcal{J}_y) = d\psi_y/dt$ being frequency of nonlinear oscillations. Let the unperturbed bunch be given by its distribution function $F(\mathcal{J}_\vartheta, \mathcal{J}_x, \mathcal{J}_z)$ normalized to unit.

II. BASIC SET OF EQUATIONS

Beam dipole moment $D_y(\vartheta, t)$ and deflecting Lorentz force $e\overline{S}_y(\vartheta, t)$ averaged over beam transverse distribution are decomposed into $\sum_k D_{y;k}, e\overline{S}_{y;k}(\Omega)\mathrm{e}^{ik\vartheta - i\Omega t}$, $y = x, z$ with Ω being the frequency of Fourier transform w.r.t. the co-rotating frame. In the laboratory frame Ω is seen as $\omega = k\omega_0 + \Omega$. Functions $f(\psi_y)$ of cyclic variables ψ_y are decomposed into Fourier series $\sum_{m_y} f_{m_y}\mathrm{e}^{im_y\psi_y}$ with m_y being the multipole index of direction $y = \vartheta, x, z$.

According to Maxwell's Eqs., the beam interacts with the vacuum chamber elements and drives horizontal deflecting S-field with

$$\overline{S}_{x;k}(\Omega) = \frac{i\beta\omega_0}{2\pi R_0} \sum_{y=x,z} Z_k^{(xy)}(k\omega_0 + \Omega) D_{y;k}(\Omega) \qquad (1)$$

where R_0 is the orbit radius, β is beam reduced velocity, $Z_k^{(xy)}(\omega)$ is the transverse (dipole) coupling impedance. Its (xy)-matrix nature accounts for the vacuum chamber cross-section anisotropy, if any. It may result in coupling of coherent motions along x, z. Here we study the standard, axisymmetric case $Z_k^{(xy)}(\omega) = Z_k(\omega)\delta_{x,y}$, $y = x, z$.

Consider a beam of average current J_0 in $M \le h$ identical and equispaced bunches, h is the main RF harmonic number, h/M is an integer. As it follows from the Vlasov's linearized Eq., the transverse BTF is

$$\begin{aligned} D_{x;k}(\Omega) &= \frac{i\pi R_0 \langle\beta_x\rangle e J_0}{2\omega_0 \beta^2 E} \times \\ &\times \sum_{k',l=-\infty}^{\infty} \delta_{k-k',lM} Y_{kk'}^{(x)}(\Omega)\overline{S}_{x;k'}(\Omega) \end{aligned} \qquad (2)$$

where $\langle\beta_x\rangle \simeq R_0\omega_0/\omega_x(0)$ is β-function averaged along the ring, E is the total energy of the beam, $\delta_{kk'}$ is the Kronecker's delta-symbol. The dispersion integrals $Y_{kk'}^{(x)}(\Omega)$ are put down in terms of multipole decomposition series

$$\begin{aligned} Y_{kk'}^{(x)}(\Omega) &= -(i\omega_0/\pi)\sum_{m_x=\pm1} m_x \sum_{m_\vartheta=-\infty}^{\infty} \times \\ &\times \iiint_0^\infty d\mathcal{J}_\vartheta d\mathcal{J}_x d\mathcal{J}_z \frac{\partial F(\mathcal{J}_\vartheta, \mathcal{J}_x, \mathcal{J}_z)}{\partial \mathcal{J}_x}\mathcal{J}_x \times \\ &\times \frac{I_{m_\vartheta,k-m_x\Delta k}(\mathcal{J}_\vartheta)\; I^*_{m_\vartheta,k'-m_x\Delta k}(\mathcal{J}_\vartheta)}{\Omega - m_\vartheta\omega_\vartheta(\mathcal{J}_\vartheta) - m_x\omega_x(\mathcal{J}_x, \mathcal{J}_z)}. \end{aligned} \qquad (3)$$

Here $\Delta k = \chi_x/\eta - \omega_x(0)/\omega_0$; $\chi_x \equiv (p_s/\omega_0)(\partial\omega_x(0)/\partial p)$ is chromaticity of the ring; $\eta = \alpha - \gamma^{-2}$, α is orbit compaction factor, γ is relativistic factor; functions $I^*_{m_\vartheta k}(\mathcal{J}_\vartheta)$ are the coefficients of series which expand a plane wave $\mathrm{e}^{ik\vartheta(\mathcal{J}_\vartheta, \psi_\vartheta)}$ into sum over longitudinal multipoles: $\sum_{m_\vartheta} I^*_{m_\vartheta k}(\mathcal{J}_\vartheta)\mathrm{e}^{im_\vartheta\psi_\vartheta}$.

Treated jointly, Eqs.1,2 yield M eigenvalue problems

$$\lambda(\Omega)D_{x;k} = R_x{}^{-1} \sum_{l'=-\infty}^{\infty} Y_{kk'}^{(x)}(\Omega) Z_{k'}(k'\omega_0 + \Omega) D_{x;k'}, \qquad (4)$$

$(k, k') = n + (l, l')M$, $-\infty < l, l' < +\infty$. Each of these stands for one of M normal coupled-bunch modes labeled by, say, $n = 0, 1, \ldots M-1$. R_x has the dimension Ohm/m of a transverse impedance,

$$R_x = -(4\beta E)\,/\,(eJ_0\langle\beta_x\rangle) < 0. \qquad (5)$$

Generally, the characteristic Eq. of coherent oscillations is

$$1 = \lambda_\ell(\Omega), \qquad \ell = (n, \ldots), \qquad (6)$$

$\lambda_\ell(\Omega)$ being an eigenvalue of Eq.4. On solving this Eq. w.r.t. Ω, one arrives at an eigenfrequency of the ℓ-th coherent mode, the unstable ones having $\mathrm{Im}\Omega > 0$.

III. A SINGLE-MODE APPROACH

To simplify the problem, we make specific the within-bunch mode subindices m_z, m_x, m_ϑ, r that follow the coupled-bunch mode index n in $\ell = (n, \ldots)$, and state conditions under which such a mode can exhibit itself solely.

1. Derivation of Eq.3 tacitly implies $m_z = 0$ which is due to the 'smooth' treatment of the uncoupled betatron x, z-oscillations. Herefrom, BTF is diagonal: $Y_{kk'}^{(xy)}(\omega) = Y_{kk'}^{(x)}(\omega)\delta_{x,y}$, $y = x, z$ (i.e., excitation by deflecting force $e\overline{S}_z$ would not drive D_x, etc).

2. Put the working point far from 2-nd order SBRs,

$$2\omega_x(\mathcal{J}_x, \mathcal{J}_z) + (m_\vartheta - m'_\vartheta)\omega_\vartheta(\mathcal{J}_\vartheta) = lM\omega_0, \qquad (7)$$

where $l = 0, 1, 2, \ldots$ ($\omega_\vartheta \ll \omega_x$); $-\infty < m_\vartheta, m'_\vartheta < +\infty$. Hence, resonant frequencies of the dipole modes $m_x = \pm 1$ would not overlap, and either can be treated separately. For definiteness, we take the upper sideband $m_x = +1$, the lower one providing no extra information on beam stability unless a SBR, Eq.7 is encountered.

3. Take bunches with a small nonlinearity,

$$|\delta\omega_\vartheta| \ll |\delta\omega_x| < \omega_\vartheta(0) \ll \omega_x(0). \qquad (8)$$

Then, at each sideband $\omega \simeq k\omega_0 + \omega_x(0) + m'_\vartheta\omega_\vartheta(0)$ near instability threshold ($\mathrm{Im}\omega \to +0$) a single resonant term whose $m_\vartheta = m'_\vartheta$ would dominate in the BTF. On dropping the rest, nonresonant items, the so called approximation of uncoupled head-tail modes m_ϑ is arrived at.

4. Assume $F(\mathcal{J}_\vartheta, \mathcal{J}_x, \mathcal{J}_z) = F_\vartheta(\mathcal{J}_\vartheta) \cdot F_{xz}(\mathcal{J}_x, \mathcal{J}_z)$. On applying to Eq.8, ignore the longitudinal tune spread, $\omega_\vartheta(\mathcal{J}_\vartheta) \simeq \omega_\vartheta(0)$. Then, characteristic Eq.6 factorizes to

$$1 = R_x^{-1}\, Y_x(\Omega)\, \zeta_r(\Omega) \qquad (9)$$

with $Y_x(\Omega)$ denoting a purely transverse dispersion integral

$$Y_x(\Omega) = -(i\omega_0/\pi)\iint_0^\infty d\mathcal{J}_x d\mathcal{J}_z \frac{\partial F_{xz}(\mathcal{J}_x, \mathcal{J}_z)}{\partial \mathcal{J}_x}\mathcal{J}_x \times$$
$$\times\ 1\,/\left((\Omega - m_\vartheta\omega_\vartheta(0)) - \omega_x(\mathcal{J}_x, \mathcal{J}_z)\right). \qquad (10)$$

Effective (instability driving) impedance $\zeta_r(\Omega)$ of mode $\ell = (n, m_z{=}0, m_x{=}1, m_\vartheta, r)$ is the r-th eigenvalue of

$$\zeta(\Omega)D_k = \sum\nolimits_{l'=-\infty}^{\infty} A_{kk'} Z_{k'}(k'\omega_0 + \Omega)D_{k'}, \qquad (11)$$

$$A_{kk'} = \int_0^\infty F_\vartheta(\mathcal{J}_\vartheta) I_{m_\vartheta, k-\Delta k}(\mathcal{J}_\vartheta) I^*_{m_\vartheta, k'-\Delta k}(\mathcal{J}_\vartheta) d\mathcal{J}_\vartheta, \qquad (12)$$

$(k, k') = n + (l, l')M$, $-\infty < l, l' < +\infty$.

5. Index r that emerges from this eigenvalue problem specifies the 'radial' (i.e., along direction $\mathcal{J}_\vartheta$ in the plane (ϑ, ϑ')) pattern of the head-tail mode m_ϑ. To ensure that only a single 'radial' mode shows itself up, consider a narrow band HOM resonance

$$Z_k(\omega) = \frac{\omega}{\omega_\varsigma} R_\varsigma \left(1 - i\frac{\omega^2 - \omega_\varsigma^2}{2\omega\Delta\omega_\varsigma}\right)^{-1} \qquad (13)$$

with coupling resistance R_ς, resonant frequency ω_ς and bandwidth $\Delta\omega_\varsigma$, the latter two complying the restrictions

$$\omega_\varsigma \not\approx lM\omega_0/2, \quad l = 1, 2, \ldots; \qquad \Delta\omega_\varsigma \ll M\omega_0. \qquad (14)$$

In this case only one ($k_1 \gtrsim -\omega_x/\omega_0$ or $k_2 \lesssim -\omega_x/\omega_0$) azimuthal harmonic of coupled-bunch mode n would fall inside the HOM bandwidth. Thus, Eq.11 reduces to

$$\zeta_r(\Omega) = A_{kk} Z_k(k\omega_0 + \Omega), \quad r = 1, \quad k = k_{1,2}, \qquad (15)$$

the unstable harmonic being k_2 (the slow betatron wave). As $\mathrm{Re}Z_k^{-1}(\omega) \simeq$ const at $\omega \simeq \pm\omega_\varsigma$, the point $R_x/\zeta_1(\Omega)$ which represents HOM's effect at $k2 = n + Ml$ in the threshold map moves almost parallel to imaginary axis of the complex plane (Y), the distance from the axis being $|R_x|/(A_{k_2k_2}R_\varsigma)$ (it does vary insignificantly due to $A_{k_2k_2}^{-1}$). Thus, the beam stability is surely guaranteed given

$$|R_x|/(A_{k_2k_2}R_\varsigma) > \Lambda_x \qquad (16)$$

where Λ_x is a maximal $\mathrm{Re}Y$-extension of threshold map,

$$\Lambda_x = \omega_0 \max_\omega \iint_0^\infty \delta\left(\omega - \omega_x(\mathcal{J}_x, \mathcal{J}_z)\right) \times \qquad (17)$$
$$\times\ \left(-\partial F_{xz}(\mathcal{J}_x, \mathcal{J}_z)/\partial \mathcal{J}_x\right) \mathcal{J}_x d\mathcal{J}_x d\mathcal{J}_z.$$

Being a sufficient stability criterion, inequality Eq.16 becomes a necessary one in large rings with $\omega_0 \lesssim \Delta\omega_\varsigma$. Up to HOM bandwidth $\Delta\omega_\varsigma$ and $\omega_\vartheta \ll \omega_x$, one can insert $k_2 \simeq -(\omega_\varsigma + \omega_x)/\omega_0$ into $A_{k_2k_2}$ to transform it into the longitudinal formfactor,

$$\Lambda_\vartheta^{(m_\vartheta)} = \int_0^\infty F_\vartheta(\mathcal{J}_\vartheta)\, |I_{m_\vartheta, -k_*}(\mathcal{J}_\vartheta)|^2\, d\mathcal{J}_\vartheta \simeq A_{k_2k_2} \qquad (18)$$

where $k_* = \omega_\varsigma/\omega_0 + \chi_x/\eta$ and $0 < \Lambda_\vartheta^{(m_\vartheta)} \le 1$. To account for all head-tail modes available, introduce the envelope

$$\Lambda_\vartheta = \max_{m_\vartheta}(\Lambda_\vartheta^{(m_\vartheta)}) \qquad (19)$$

which is a function of the external parameters only: $F_\vartheta(\mathcal{J}_\vartheta)$, $\omega_\varsigma/\omega_0$, χ_x/η. On adopting the above assumptions, one finally arrives at the stability criterion

$$R_\varsigma \le \frac{|R_x|}{\Lambda_\vartheta\ \Lambda_x} = \frac{1}{\Lambda_\vartheta\ \Lambda_x} \times \frac{4\,\beta E}{eJ_0\langle\beta_x\rangle} \qquad (20)$$

with two bunch formfactors Λ_ϑ, Λ_x left to be estimated.

IV. FORMFACTORS

A. Longitudinal Formfactor

According to Eq.8, $|\delta\omega_\vartheta| \ll \omega_\vartheta(0)$ and the law of motion along ϑ is just $\vartheta(\mathcal{J}_\vartheta, \psi_\vartheta) \simeq \sqrt{\mathcal{J}}\cos(\psi_\vartheta + \psi_{\vartheta 0})$. Hence,

$$|I_{m_\vartheta k}(\mathcal{J}_\vartheta)|^2 \simeq J_{m_\vartheta}^2\left(k\Delta\vartheta_0\sqrt{\mathcal{J}_\vartheta/\mathcal{J}_{\vartheta 0}}\right) \qquad (21)$$

with $J_m(y)$ denoting Bessel functions of the m-th order, $\Delta\vartheta_0 = \Delta\vartheta(\mathcal{J}_{\vartheta 0})$ being longitudinal half-width of the bunch (in other words, oscillation amplitude along ϑ at a phase-plane trajectory $\mathcal{J}_\vartheta = \mathcal{J}_{\vartheta 0}$). It implies the following reflection properties

$$\Lambda_\vartheta^{(-m_\vartheta)} = \Lambda_\vartheta^{(m_\vartheta)};\ \Lambda_\vartheta^{(m_\vartheta)}(-k_*\Delta\vartheta_0) = \Lambda_\vartheta^{(m_\vartheta)}(k_*\Delta\vartheta_0). \quad (22)$$

Globally, formfactor $\Lambda_\vartheta^{(0)}$ of the rigid-bunch head-tail mode $m_\vartheta = 0$ dominates, envelope Λ_ϑ, Eq.19 thus coinciding with $\Lambda_\vartheta^{(0)}$ (except for a small region near $|k_*\Delta\vartheta_0| \simeq$ 3–5 where mode $|m_\vartheta| = 1$ may exhibit itself).

Replace $J_m^2(y)$ in Eq.18 by their quadratic small-argument and trigonometric large-argument (with 1/2 substituted for $\cos^2(\ldots)$) asymptotes. On integrating, one obtains with accuracy sufficient for practical purposes,

$$\Lambda_\vartheta \simeq \Lambda_\vartheta^{(0)} \simeq \begin{cases} 1 - \frac{1}{2}\left\langle \theta^2 \right\rangle |k_*\Delta\vartheta_0|^2, & |k_*\Delta\vartheta_0| \lesssim 2; \\ \frac{1}{\pi}\left\langle \theta^{-1} \right\rangle |k_*\Delta\vartheta_0|^{-1}, & |k_*\Delta\vartheta_0| \gtrsim 3. \end{cases} \quad (23)$$

Here, numerical factors $\left\langle \theta^2 \right\rangle \leq 1$ and $\left\langle \theta^{-1} \right\rangle \geq 1$ with $\theta = \vartheta/\Delta\vartheta_0$ are, respectively, mean-square and mean-reciprocal reduced half-widths of a bunch,

$$\begin{matrix} \left\langle \theta^2 \right\rangle \\ \left\langle \theta^{-1} \right\rangle \end{matrix} = \int_0^\infty \begin{matrix} (\mathcal{J}_\vartheta/\mathcal{J}_{\vartheta 0}) \\ (\mathcal{J}_\vartheta/\mathcal{J}_{\vartheta 0})^{-1/2} \end{matrix}\ F_\vartheta(\mathcal{J}_\vartheta)\, d\mathcal{J}_\vartheta. \quad (24)$$

B. Transverse Formfactor

Let us introduce normalized to unit 1-D transverse distributions $F_x(\mathcal{J}_x)$ and $F_z(\mathcal{J}_z)$ where, say, $F_x(\mathcal{J}_x)$ is

$$F_x(\mathcal{J}_x) = \int_0^\infty F_{xz}(\mathcal{J}_x, \mathcal{J}_z)\, d\mathcal{J}_z. \quad (25)$$

Take into account the cubic nonlinearity of the magnetic field which results in betatron tune spread

$$\omega_x(\mathcal{J}_x, \mathcal{J}_z) \simeq \omega_x(0) + \frac{\partial\omega_x}{\partial\mathcal{J}_x}(0)\mathcal{J}_x + \frac{\partial\omega_x}{\partial\mathcal{J}_z}(0)\mathcal{J}_z, \quad (26)$$

coefficients at $\mathcal{J}_x$ and $\mathcal{J}_z$ being controlled with the octupole correctors.

Formfactor Λ_x is amenable to straightforward calculations in two particular cases. Indeed, for $\partial\omega_x/\partial\mathcal{J}_z = 0$

$$\Lambda_x = \frac{b_{xx}}{|\delta\omega_{xx}/\omega_0|}, \quad \delta\omega_{xx} = \frac{\partial\omega_x}{\partial\mathcal{J}_x}(0)\,\mathcal{J}_{x0}, \quad (27)$$

$$b_{xx} = \mathcal{J}_{x0} \max_{\mathcal{J}_x \geq 0}\left(\mathcal{J}_x\left(-\partial F_x(\mathcal{J}_x)/\partial\mathcal{J}_x\right)\right). \quad (28)$$

On the other hand, for $\partial\omega_x/\partial\mathcal{J}_x = 0$ it follows that

$$\Lambda_x = \frac{b_{xz}}{|\delta\omega_{xz}/\omega_0|}, \quad \delta\omega_{xz} = \frac{\partial\omega_x}{\partial\mathcal{J}_z}(0)\,\mathcal{J}_{z0}, \quad (29)$$

$$b_{xz} = \mathcal{J}_{z0} \max_{\mathcal{J}_z \geq 0}\left(F_z(\mathcal{J}_z)\right) = \mathcal{J}_{z0}\, F_z(0). \quad (30)$$

Here $\mathcal{J}_{x0}$, $\mathcal{J}_{z0}$ are the action variables at the (effective) edge of the bunch; $\delta\omega_{xx}$, $\delta\omega_{xz}$ are the partial betatron tune spreads, both having an arbitrary sign.

On inserting Eq.26 into Eq.17, one sees that Λ_x is kept intact by a simultaneous reversal of signs in $\delta\omega_{xx}$ and $\delta\omega_{xz}$. Therefore, taking into account the exact Eqs.27–30 and inflicting no loss to generality, rewrite Λ_x as

$$\Lambda_x = f_x\left(\frac{\delta\omega_{xx}}{\delta\omega_{xz}};\ldots\right) \times \left(\left(\frac{\delta\omega_{xx}}{\omega_0 b_{xx}}\right)^2 + \left(\frac{\delta\omega_{xz}}{\omega_0 b_{xz}}\right)^2\right)^{-1/2}, \quad (31)$$

$$f_x(\pm\infty;\ldots) = f_x(0;\ldots) = 1.$$

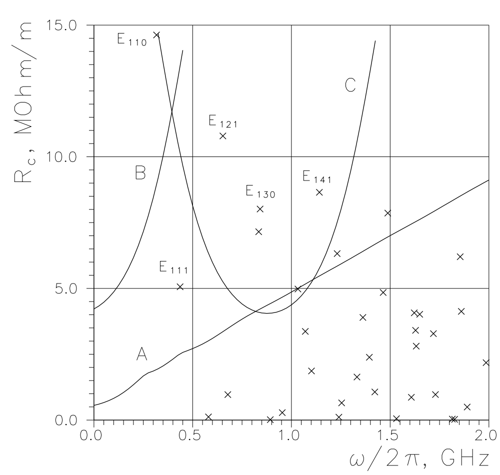
Dots in f_x show its dependence on details of joint distribution $F_{xz}(\mathcal{J}_x, \mathcal{J}_z)$. Fortunately, the calculations show that f_x is rather insensitive to $\delta\omega_{xx}/\delta\omega_{xz}$ for realistic distributions. With a good accuracy Eq.31 can be used with $f_x \simeq 1$, which plainly puts down transverse formfactor as a reciprocal of an effective betatron tune spread,

$$\Lambda_x \simeq \left(\left(\frac{\delta\omega_{xx}}{\omega_0 b_{xx}}\right)^2 + \left(\frac{\delta\omega_{xz}}{\omega_0 b_{xz}}\right)^2\right)^{-1/2}. \quad (32)$$

Eqs.20, 23, 32 are the sought-for tool for practical estimates of head-tail instability thresholds.

V. EXAMPLE OF APPLICATION

Consider the UNK 1-st Stage which is to be equipped with $N = 8 \times 2 = 16$ conventional copper cavities, their length being L = 0.5 m; radius r_0 = 0.577 m; surface resistance σ^{-1} = 1.7·10^{-8} Ohm·m. The figure shows coupling impedances per one cavity for dipole HOMs E_{1np}. Tolerable values of R_ς are found with Eqs.20,23,32; J_0 = 1.4 A; α = 4.95·10^{-4}; ω_x/ω_0 = 55.7; $\delta\omega_{xx}/\omega_0$ = $\delta\omega_{xz}/\omega_0$ = 0.5·10^{-2}; $\langle\beta_x\rangle$ = 93.5 m. Curve A: injection at E = 65 GeV with $h\Delta\vartheta_0/\pi$ = 0.54 and standard $\chi_x \simeq +3$. Curve B: the same for $\chi_x \simeq +3$ at E = 600 GeV, $h\Delta\vartheta_0/\pi = 0.38$. Curve C: large negative $\chi_x \simeq -30$ as required by a slow extraction scheme.



Evidently, at least nine of the UNK cavity transverse HOMs are to be damped with a dedicated system.

More details on the topic can be found in Ref.[1].

References

[1] S. Ivanov, M. Pozdeev, IHEP Preprint 94–110, Protvino, 1994 (in Russian).