THE COUPLING IMPEDANCE OF A TOROIDAL BEAM PIPE WITH CIRCULAR CROSS SECTION*

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Abstract

An analytic expression for the non-resonant longitudinal coupling impedance in a toroidal beam pipe with circular cross section is derived using a perturbation treatment carried out in a local orthogonal coordinate system.

I. INTRODUCTION

In this paper, the topic of coupling impedances and beam-induced forces in a toroidal beam pipe is revisited. In straight, smooth accelerator beam pipe configurations, the space charge forces on a charged test particle due to beam-induced electric and magnetic fields are subject to a near perfect cancellation in the ultra-relativistic limit. It is well known that in toroidal geometries the cancellation is imperfect, resulting in residual, "energy-independent" longitudinal and transverse forces which can impact the accelerator performance. Whereas the present-day designs of high-energy hadron accelerators/colliders are based on beam pipes with circular cross section, essentially all theoretical studies assume rectangular geometries, the notable exception being the report by Zotter.[1] The expressions here presented are intended to amend this deficiency.

The primary objective of this paper is the derivation of an expression for the residual longitudinal Z/n of a toroidal beam pipe with circular cross section, with b representing the beam pipe radius and R the curvature radius of the central arc as shown in Fig. 1. The results are obtained via the perturbation method developed by Jouguet[2] for the analysis of the electromagnetic wave propagation in curved waveguides, which recently has been successfully applied to the derivation of expressions for the longitudinal coupling impedance in toroidal beam pipes with rectangular cross section.[3,4] Using the Serret-Frenet frame, an appropriate "local" orthogonal coordinate system $(r,\ \varphi,\ \theta)$ can be errected around the central arc of the torus. This choice is superior to the use of toroidal coordinates, since the local coordinate system reduces to the usual circular-cylinder coordinates $r,\ \varphi,\ s = R\theta$ as required for the perturbation treatment of the problem. Since the residual longitudinal coupling impedance of an on-axis beam must be independent of the direction of curvature, an expansion to second order in $1/R$ is required. It is known from previous studies that the residual coupling impedance does not exhibit a longarithmic divergence, if the transverse beam size is reduced to zero. Consequently, the study can be limited to filamentary beams. Since wall losses are themselves a small perturbation, they can be ignored in deriving the perturbation results due to the curvature.

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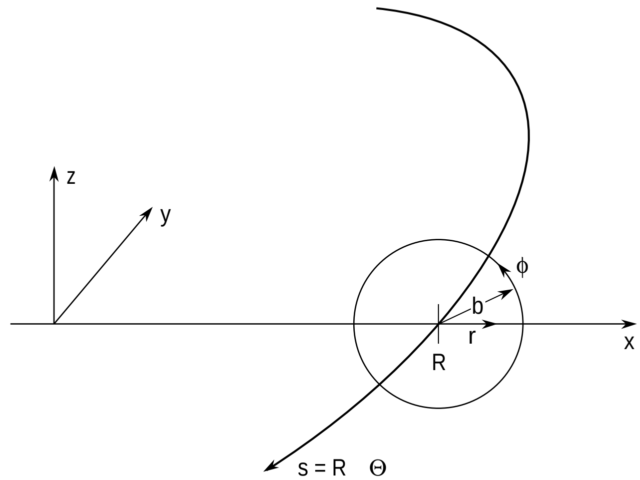

Figure 1. Local coordinate system (r, φ, θ) for toroidal beam tube with circular cross section.

II. PERTURBATIVE SOLUTION OF MAXWELL'S EQUATIONS

The "local" orthogonal curvilinear coordinate system r, φ, θ appropriate to the toroidal beam pipe geometry is defined in terms of the Cartesian coordinates x, y, z

$$\begin{aligned} x &= (R + r\cos\varphi)\cos\theta \\ y &= -(R + r\cos\varphi)\sin\theta \\ z &= r\sin\varphi \end{aligned} \tag{1}$$

Assuming time harmonic fields of the general form $\mathcal{F}(r,\ \varphi)e^{-jn\theta}e^{j\omega t}$ with $\omega = v\nu$ and $\nu = n/R$ (in natural units where $c = 1,\ \mu_o = 1$) one can write Maxwell's equations in the source free regions as

$$\begin{aligned} \frac{\partial g\mathcal{E}_\theta}{r\partial r} + j\nu\mathcal{E}_\varphi &= -j\omega g\mathcal{H}_r \\ j\nu\mathcal{E}_r + \frac{\partial g\mathcal{E}_\theta}{\partial r} &= j\omega g\mathcal{H}_\varphi \\ \frac{\partial r\mathcal{E}_\varphi}{r\partial r} - \frac{\partial \mathcal{E}_r}{r\partial\varphi} &= -j\omega\mathcal{H}_\theta \\ \frac{\partial g\mathcal{H}_\theta}{r\partial\varphi} + j\nu\mathcal{H}_\varphi &= j\omega g\mathcal{E}_r \\ j\nu\mathcal{H}_r + \frac{\partial g\mathcal{H}_\theta}{\partial r} &= -j\omega g\mathcal{E}_\varphi \\ \frac{\partial r\mathcal{H}_\varphi}{r\partial r} - \frac{\partial\mathcal{H}_r}{r\partial\varphi} &= j\omega\mathcal{E}_\theta \end{aligned} \tag{2}$$

with $g = 1 + (r/R)\cos\varphi$.

Perturbative solutions can be found by expanding the fields in inverse powers of the curvature radius, or explicitely

$$\mathcal{E}_{r,\varphi} = E_{r,\varphi} + \frac{1}{R}e_{r,\varphi} + \cdots$$

$$\begin{aligned}
\mathcal{E}_\theta &= j\left(E_\theta + \frac{1}{R}e_\theta + \cdots\right) \\
\mathcal{H}_{r,\varphi} &= H_{r,\varphi} + \frac{1}{R}h_{r,\varphi} + \cdots \\
\mathcal{H}_\theta &= j\left(H_\theta + \frac{1}{R}h_\theta + \cdots\right) \qquad (3)
\end{aligned}$$

The resulting expressions for Maxwell's equations to first order in R^{-1} are given as follows

$$\begin{aligned}
&\frac{1}{r}\frac{\partial e_\theta}{\partial\varphi} + \nu e_\varphi + \omega h_r = -\frac{\partial}{\partial\varphi}(\cos\varphi E_\theta) - \omega r\cos\varphi H_r \\
&\nu e_r + \frac{\partial e_\theta}{\partial r} - \omega h_\varphi = -\cos\varphi\frac{\partial}{\partial r}(rE_\theta) + \omega r\cos\varphi H_\varphi \\
&\frac{\partial}{\partial r}(re_\varphi) - \frac{\partial e_r}{\partial\varphi} - \omega r h_\theta = 0 \\
&\frac{1}{r}\frac{\partial h_\theta}{\partial\varphi} + \nu h_\varphi - \omega e_r = -\frac{\partial}{\partial\varphi}(\cos\varphi H_\theta) + \omega r\cos\varphi E_r \\
&\nu h_r + \frac{\partial h_\theta}{\partial r} + \omega e_\varphi = -\cos\varphi\frac{\partial}{\partial r}(rH_\theta) - \omega r\cos\varphi E_\varphi \\
&\frac{\partial}{\partial r}(rh_\varphi) - \frac{\partial h_r}{\partial\varphi} + \omega r e_\theta = 0 \qquad (4)
\end{aligned}$$

Decoupling Maxwell's equations and reducing them to two independent partial differential equations in the azimuthal components e_θ and h_θ can be achieved by introducing complex transverse fields and differential operators.[5]

After some manipulations, one finds the independent differential equations for the azimuthal perturbations

$$\begin{aligned}
\Delta_T e_\theta - \kappa^2 e_\theta &= -\Delta_T(r\cos\varphi E_\theta) \\
&\quad + \frac{1}{r}\frac{\partial}{\partial r}\left\{r^2\cos\varphi(\nu E_r + \omega H_\varphi)\right\} \\
&\quad + \frac{\partial}{\partial\varphi}\left\{\cos\varphi(\nu E_\varphi - \omega H_r)\right\} \\
\Delta_T h_\theta - \kappa^2 h_\theta &= -\Delta_T(r\cos\varphi H_\theta) \\
&\quad + \frac{1}{r}\frac{\partial}{\partial r}\left\{r^2\cos\varphi(\nu H_r - \omega E_\varphi)\right\} \\
&\quad + \frac{\partial}{\partial\varphi}\left\{\cos\varphi(\nu H_\varphi + \omega E_r)\right\} \qquad (5)
\end{aligned}$$

with $\kappa^2 = \nu^2 - \omega^2 = (1 - v^2)\nu^2 = (\nu/\gamma)^2$ and the transverse Laplacian operator

$$\Delta_T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

The solution is uniquely determined by imposing boundary and continuity conditions. Having determined e_θ and h_θ, the remaining transverse components can be obtained as follows

$$\begin{aligned}
\kappa^2 e_r &= -\nu\frac{\partial e_\theta}{\partial r} - \nu\cos\varphi\frac{\partial rE_\theta}{\partial r} + \omega^2 r\cos\varphi E_r \\
&\quad - \frac{\omega}{r}\frac{\partial h_\theta}{\partial\varphi} - \omega\frac{\partial}{\partial\varphi}\cos\varphi H_\theta + \omega\nu r\cos\varphi H_\varphi \\
\kappa^2 h_r &= -\nu\frac{\partial h_\theta}{\partial r} - \nu\cos\varphi\frac{\partial rH_\theta}{\partial r} + \omega^2 r\cos\varphi H_r \\
&\quad + \frac{\omega}{r}\frac{\partial e_\theta}{\partial\varphi} + \omega\frac{\partial}{\partial\varphi}\cos\varphi E_\theta - \omega\nu r\cos\varphi E_\varphi \\
\kappa^2 e_\varphi &= -\frac{\nu}{r}\frac{\partial e_\theta}{\partial\varphi} - \nu\frac{\partial}{\partial\varphi}\cos\varphi E_\theta + \omega^2 r\cos\varphi E_\varphi \\
&\quad + \omega\frac{\partial h_\theta}{\partial r} + \omega\cos\varphi\frac{\partial rH_\theta}{\partial r} - \omega\nu r\cos\varphi H_r \\
\kappa^2 h_\varphi &= -\frac{\nu}{r}\frac{\partial h_\theta}{\partial\varphi} - \nu\frac{\partial}{\partial\varphi}\cos\varphi H_\theta + \omega^2 r\cos\varphi H_\varphi \\
&\quad - \omega\frac{\partial e_\theta}{\partial r} - \omega\cos\varphi\frac{\partial rE_\theta}{\partial r} + \omega\nu r\cos\varphi E_r \qquad (6)
\end{aligned}$$

The internal consistency of the results can be checked by testing for a divergence free solution using div $\vec{E} = 0$:

$$\begin{aligned}
\frac{\partial re_r}{\partial r} + \frac{\partial e_\varphi}{\partial\varphi} + \nu re_\theta &= -r\cos\varphi\left(E_r + \frac{\partial rE_r}{\partial r}\right) \\
&\quad - r\frac{\partial\cos\varphi E_\varphi}{\partial\varphi} \qquad (7)
\end{aligned}$$

and div $\vec{H} = 0$:

$$\begin{aligned}
\frac{\partial rh_r}{\partial r} + \frac{\partial h_\varphi}{\partial\varphi} + \nu rh_\theta &= -r\cos\varphi\left(H_r + \frac{\partial rH_r}{\partial r}\right) \\
&\quad - r\frac{\partial\cos\varphi H_\varphi}{\partial\varphi} \qquad (8)
\end{aligned}$$

III. THE CURVATURE-INDUCED RESIDUAL COUPLING IMPEDANCE

First order perturbation results in R^{-1} are the required first step towards finding the curvature-induced longitudinal coupling impedance. In order to prevent a logarithmic divergence of the result, the beam must be given a finite transverse size. Solving the case of a tubular beam located at the radius ρ avoids the divergence. The tubular beam is assumed to travel in the θ direction with velocity v and has the current density

$$i_\theta = \frac{I}{2\pi\rho}\delta(r-\rho)e^{-jn\theta}e^{j\omega t} \qquad (9)$$

Perturbative solutions to Maxwell's equations are found as described above separately for inner $(r < \rho)$ and outer region $(\rho < r < b)$ and by field matching at $r = \rho$, while imposing Ampere's law on H_φ and satisfying the boundary condition at $r = b$, assuming lossless walls.

The field components in the limit of filamentary $(\rho \to 0)$ and ultra-relativistic $(v \to 1)$ beams required in the derivation of $\mathcal{E}_\theta$ to second order can be written as

$$\begin{aligned}
\mathcal{E}_r &= \frac{I}{2\pi r} + \frac{1}{R}e_{r1}\cos\varphi + \frac{1}{R^2}\cdots \\
\mathcal{E}_\theta &= j\left\{\frac{1}{R}e_{\theta 1}\cos\varphi + \frac{1}{R^2}\left(e_{\theta 2} + e_{\theta 22}\cos 2\varphi\right) + \frac{1}{R^3}\cdots\right\} \\
\mathcal{H}_\varphi &= \frac{I}{2\pi r} + \frac{1}{R}h_{\varphi 1}\cos\varphi + \frac{1}{R^2}\cdots \qquad (10)
\end{aligned}$$

with

$$\begin{aligned}
e_{\theta 1} &= -\frac{I}{2\pi}\nu r\ln\frac{b}{r} \qquad (11) \\
e_{r1} &= -\frac{I}{2\pi}\left\{\frac{1}{2} - \frac{1}{2}\ln\frac{b}{r} - \frac{\nu^2 b^2}{16}\left(7 - \frac{r^2}{b^2} + 4\frac{r^2}{b^2}\ln\frac{b}{r}\right)\right\} \\
h_{\theta 1} &= -\frac{I}{2\pi}\left\{\frac{1}{2} + \frac{1}{2}\ln\frac{b}{r} - \frac{\nu^2 b^2}{16}\left(7 - \frac{r^2}{b^2} + 4\frac{r^2}{b^2}\ln\frac{b}{r}\right)\right\}
\end{aligned}$$

The second order field perturbations of an on-axis filamentary beam are excited by first-order perturbations with pure dipole, i.e. 1φ-dependence. The factor $\cos^2\varphi = \frac{1}{2} + \frac{1}{2}\cos 2\varphi$ in the forcing term implies that the second order fields have only φ independent and quadrupole, i.e. 2φ, dependent terms. Only the φ-independent term $e_{\theta 2}$ leads to an on-axis electric field component in θ-direction which is responsible for the residual coupling impedance. The differential equation for the relevant φ-independent $e_{\theta 2}$ component thus reduces in the ultra-relativistic limit, i.e. $\kappa = 0$, to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial e_{\theta 2}}{\partial r}\right) = -\frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial r e_{\theta 1}}{\partial r}\right) + \frac{\nu}{2r}\frac{\partial}{\partial r}\left(r^2(e_{r1}+h_{\varphi 1})\right) \tag{12}$$

together with the boundary conditions on the beam pipe wall $(e_{\theta 2})_{r=b} = 0$ and, replacing the matching condition, $(\partial e_{\theta 2}/\partial r)_{r=0} = 0$.

The solution is found to be

$$\begin{aligned} e_{\theta 2} &= \frac{I}{2\pi}\frac{\nu b^2}{4}\left\{\left(1-\frac{7}{8}\nu^2 b^2\right)\left(1-\frac{r^2}{b^2}\right)\right. \\ &\qquad \left. +2\left(1+\frac{\nu^2 r^2}{8}\right)\frac{r^2}{b^2}\ln\frac{b}{r}\right\} \end{aligned} \tag{13}$$

from which follow the expression for the residual longitudinal coupling impedance seen by a filamentary beam in the ultra-relativistic case, $\gamma \to \infty$ ($Z_o = c\mu_o$, in SI units)

$$\frac{Z}{n} = -j\frac{b^2 Z_o}{4R^2}\left(1-\frac{7}{8}\nu^2 b^2\right) \tag{14}$$

The full second-order expression for the curvature-induced longitudinal coupling impedance and mathematical details of its derivation can be found in a journal paper,[6] where the impedance in the long-wavelength limit is given as

$$\begin{aligned} \frac{Z}{n} &= -j\beta Z_0\frac{b^2}{4R^2}\left(1-\frac{7}{8}\nu^2 b^2\right) \\ &\quad +j\frac{Z_0}{16\beta\gamma^2}\frac{b^2}{R^2}\left(1-2\nu^2 b^2+\frac{109}{48}\nu^4 b^4\right) \end{aligned} \tag{15}$$

with $\beta = v/c$.

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