REVIEW OF LONGITUDINAL PERTURBATION FORMALISM*

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I. INTRODUCTION

The foundation of the beam perturbation formalism was established on the Sacherer integral equation [1]. The original solution was obtained by using simplified radial functions, and the results can be used to explain several important beam instability problems. To solve the instabilities caused by high beam intensities calls for more complete solution and the extension to the formalism as well. For the non-coupled perturbation problem without frequency spread, two different eigenvalue type solutions were proposed. The formalism is also modified to solve the azimuthal mode coupling instability and the beam instability with synchrotron frequency spread, where both dispersion type and eigenvalue type solutions were proposed. In this article, a brief review of longitudinal perturbation formalism based on Sacherer integral equation is given. Some perspectives will also be discussed.

II. SACHERER INTEGRAL EQUATION

At the low intensity regime, the azimuthal mode coupling can be neglected, and the Sacherer integral equation is written as [2-4],

$$(\omega - m\omega_s)\, R^{(m)}(r) = jm\omega_s \xi\, W(r) \times \sum_{p=-\infty}^{\infty} \frac{Z(p)}{p}\, J_m(pr)\Lambda^{(m)}(p) \qquad (1)$$

where ω is the coherent frequency shift, m is the azimuthal mode number, ω_s is the synchrotron frequency, and $R^{(m)}(r)$ is the radial function defined on the amplitude of oscillation, r. The scaling factor ξ is defined as $\xi = 2\pi I_o/(V\cos\phi_s)$, where I_o is the average beam current, ϕ_s is the synchronous phase and V is the total RF voltage. The weight function is defined as $W(r) = -d\psi_o/(dr\cdot r)$, where ψ_o is the stationary particle distribution. Z(p) in (1) is the impedance with p representing the frequency samplings. The Hankel spectrum $\Lambda^{(m)}(p)$ is related to the radial function by,

$$\Lambda^{(m)}(p) = \int_o^{\infty} R^{(m)}(r) J_m(pr) r dr \qquad (2)$$

where $J_m(pr)$ is the mth order Bessel function.

2.1 Sacherer's Solution

The equation (1) is an eigenvalue problem, which has two unknown variables ω and $R^{(m)}(r)$, and therefore, cannot be solved straightforwardly. To get approximate solutions, Sacherer assumed simplified modes, for example sinusoidal line densities, and solved the equation [1]. The result is very useful in many applications. For high intensity applications, more exact solution is needed, which appears to be in two different forms of eigenvalue problems.

2.2 Solution Using Orthogonal Polynomials

Using orthogonal polynomial expansion of radial modes is a conventional approach in treating an integral equation such as the Sacherer's. For the weight function *W(r)*, a set of normalized orthogonal polynomials $f_k^{(m)}(r)$ can always be found. The radial function can be expanded on these polynomials as,

$$R^{(m)}(r) = W(r) \sum_{K'=0}^{\infty} \alpha_{k'}^{(m)} f_{k'}^{(m)}(r) \qquad (3)$$

and the Bessel function $J_m(pr)$ can be written as,

$$J_m(pr) = \sum_{k'=0}^{\infty} \Lambda_{k'}^{(m)}(p) f_{k'}^{(m)}(r) \qquad (4)$$

where the Hankel spectrum of the orthogonal polynomial is,

$$\Lambda_k^{(m)}(p) = \int_0^{\infty} W(r) f_k^{(m)}(r) J_m(pr) r dr \qquad (5)$$

After some manipulations, the equation (1)can be written as an eigenvalue problem [4],

$$(\omega - m\omega_s)\, \boldsymbol{\alpha}^{(m)} = \boldsymbol{M}^{(m)} \boldsymbol{\alpha}^{(m)} \qquad (6)$$

with the eigenvector $\boldsymbol{\alpha}^{(m)} = [\alpha_0^{(m)} \ldots \alpha_{\bar{k}}^{(m)}]^T$ where $\bar{k}$ denotes truncation and the superscript *T* denotes transpose. The *k,l*th element of the system matrix $\mathbf{M}^{(m)}$ is,

$$M_k^{(m)} = jm\omega_s \xi \sum_{p=-\infty}^{\infty} \frac{Z(p)}{p}\, \Lambda_k^{(m)}(p) \Lambda_l^{(m)}(p) \qquad (7)$$

In solving $|(\omega - m\omega_S)\, \mathbf{I} - \mathbf{M}^{(m)}| = 0$, the eigenvalues can be found. A similarity transformation can be applied to the matrix $\mathbf{M}^{(m)}$, which yields both the eigenvalues and eigenvectors simultaneously. If we take only the first orthogonal polynomial, then (6) becomes a scalar equation. The solution obtained under the conditions such as a long bunch and narrow-band impedances is comparable to Sacherer' solution using the simplified model, with only small differences.

2.3 Solution Using Handel Samplings

Another eigenvalue type of formalism in solving (1) is as follows [5],

$$(\omega - m\omega_s)\, \boldsymbol{\Lambda}_P^{(m)} = \boldsymbol{K}^{(m)} \boldsymbol{\Lambda}_P^{(m)} \qquad (8)$$

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where $\mathbf{\Lambda}_P^{(m)} = [\Lambda^{(m)}(-\bar{p}) \ldots \Lambda^{(m)}(\bar{p})]^T$ represents Hankel harmonic samplings, and the p,qth element of the system matrix $\mathbf{K}^{(m)}$ is,

$$K_{p,q}^{(m)} = jm\omega_s\xi \frac{Z(p)}{p}\int_0^\infty W(r)J_m(pr)J_m(qr)rdr \qquad (9)$$

To find the relation between the two approaches, we write the impedance matrix as,

$$\boldsymbol{Z} = diag\{ \frac{Z(-\bar{p})}{-\bar{p}} \quad \frac{Z(-\bar{p}+1)}{-\bar{p}+1} \ldots \frac{Z(\bar{p})}{\bar{p}} \} \qquad (10)$$

and the Hankel spectrum matrix,

$$\mathbf{\Lambda}^{(m)} = \begin{bmatrix} \Lambda_0^{(m)}(-\bar{p}) & \Lambda_0^{(m)}(-\bar{p}+1) & \ldots & \Lambda_0^{(m)}(\bar{p}) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \Lambda_k^{(m)}(-\bar{p}) & \Lambda_k^{(m)}(-\bar{p}+1) & \ldots & \Lambda_K^{(m)}(\bar{p}) \end{bmatrix} \qquad (11)$$

then we get,

$$\boldsymbol{M}^{(m)} = jm\omega_s\xi\mathbf{\Lambda}^{(m)}\boldsymbol{Z}\mathbf{\Lambda}^{(m)T} \qquad (12)$$

$$\boldsymbol{K}^{(m)} = jm\omega_s\xi\mathbf{\Lambda}^{(m)T}\mathbf{\Lambda}^{(m)}\boldsymbol{Z} \qquad (13)$$

Note that we have $\mathbf{\Lambda}_P^{(m)} = \mathbf{\Lambda}^{(m)T}\boldsymbol{\alpha}^{(m)}$. By left multiplying $\mathbf{\Lambda}^{(m)T}$ to (6), we have,

$$(\omega - m\omega_s)\mathbf{\Lambda}_P^{(m)} = \mathbf{\Lambda}^{(m)T}\boldsymbol{M}^{(m)}\boldsymbol{\alpha}^{(m)} = \boldsymbol{K}^{(m)}\mathbf{\Lambda}_P^{(m)} \qquad (14)$$

which is the same as (8).

To compare the two different eigenvalue problems, we note that the dimension of $\mathbf{M}^{(m)}$ is determined by the number of orthogonal polynomials, and the dimension of $\mathbf{K}^{(m)}$ is determined by the frequency harmonic number. In general, the dimension of the latter is larger than the former, but the computational load is about the same. The calculation of $K_{p,q}^{(m)}$ of (9) does not require the truncation in *k*, and therefore it can be more accurate. On the other hand, sometimes $M_{k,l}^{(m)}$ can be directly calculated on the line densities, avoiding the truncation on *p* [4].

III. MODE COUPLING

When the beam intensity increases, the coupling between the azimuthal modes can no longer be neglected. The equation (1) becomes,

$$(\omega - m\omega_s)R^{(m)}(r) = j^{m+1}m\omega_S\xi W(r) \times \sum_{p=-\infty}^{\infty} \frac{Z(p)}{p} J_m(pr) \sum_{m'=-\infty}^{\infty} j^{-m'}\Lambda^{m'}(p) \qquad (15)$$

This equation can be solved as either an eigenvalue problem [6] or a dispersion relation problem [3].

3.1 Eigenvalue Type Solution

To be not overwhelmed by large dimensions, we only consider the coupling between *m* and *m'*,which can be easily extended to include any and all necessary modes. Using orthogonal polynomial expansion, the following equation is obtained.

$$\begin{bmatrix} (\omega - m\omega_s)\boldsymbol{I} & 0 \\ 0 & (\omega - m'\omega_s)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}^{(m)} \\ \boldsymbol{\alpha}^{(m')} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}^{(m,m)} & \boldsymbol{M}^{(m,m')} \\ \boldsymbol{M}^{(m',m)} & \boldsymbol{M}^{(m',m')} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}^{(m)} \\ \boldsymbol{\alpha}^{(m')} \end{bmatrix} \qquad (16)$$

where the *k*,*l*th element in $\mathbf{M}^{(m,m')}$ is,

$$M_{k,l}^{(m,m')} = j^{m+1-m'}m\omega_s\xi \times \sum_{p=-\infty}^{\infty} \frac{Z(p)}{p} \Lambda_k^{(m)}(p)\Lambda_l^{(m')}(p) \qquad (17)$$

The equation (16) is formally an eigenvalue problem. It is noted that in general the orthogonality between the polynomials of the different azimuthal modes is not guaranteed. Since the polynomials of the azimuthal mode *m* is further modulated by the rotation factor $e^{jm\theta}$, which itself is orthogonal for *m*, the orthogonality between the polynomials in different azimuthal modes is implicitly implied, and therefore (16) can be treated as an eigenvalue problem.

The coupling between the modes m = ±1 in fact is the same problem the conventional Robinson approach han-dled, which leads to the well known second Robinson criterion. The following is a brief comparison between the two approaches. In the Robinson approach, only the fundamental modes are concerned, the impedances are considered in a form of second order transfer function, and the particle distribution is only reflected in the ratio of the beam DC current and the fundamental current. In the mode coupling approach, more than fundamental modes are included, the impedances are considered with the real and imaginary parts, and the particle distribution is used in the calculation. Examples can be found to show that the mode coupling is a more reliable treatment of the problem.

3.2 Dispersion relation Type Solution

The following equation is derived from (15), similarly to (8),

$$(\omega - m\omega_s)\, j^{-m}\mathbf{\Lambda}_P^{(m)} = \boldsymbol{K}^m \sum_{m'=-\infty}^{\infty} j^{-m'}\mathbf{\Lambda}_P^{(m')} \qquad (18)$$

Dividing (18) by $\omega - m\omega_s$ and summing over *m*, we get an equation for the azimuthal mode coupling,

$$\sum_{m=-\infty}^{\infty} j^{-m}\mathbf{\Lambda}_P^{(m)} = \sum_{m=-\infty}^{\infty} \boldsymbol{K}^{(m)}/(\omega - m\omega_s) \sum_{m'=-\infty}^{\infty} j^{-m'}\,\mathbf{\Lambda}_P^{(m')} \qquad (19)$$

Because of the denominator on the right side of the equation, it is clear that (19) is a dispersion relation type equation. To find the relation between the dispersion equation (19) and the eigenvalue problem(16), we move the factor $(\omega - m\omega_s)$ in the equation (16) to the right side, then multiply from the left by $[j^{-m}\mathbf{\Lambda}^{(m)T}\, j^{-m'}\mathbf{\Lambda}^{(m')T}]$.

IV. FREQUENCY SPREAD EFFECT

Taking the synchrotron frequency spread into consideration, we write $\omega_s(r) = \omega_{sc} + D(r)$, where ω_{sc} is the synchrotron frequency at the beam center, and the frequency dispersion function *D(r)* represents the dependence of the synchrotron frequency on the amplitude of oscillation.

4.1 Eigenvalue Type Solution

Substituting $\omega_s(r)$ into (1), moving $D(r)R^{(m)}(r)$ to the right side, and using orthogonal polynomial expansion, we get [2,7],

$$(\omega - m\omega_{sc})\boldsymbol{\alpha}^{(m)} = (m\mathbf{N}^{(m)} + \mathbf{M}^{(m)})\boldsymbol{\alpha}^{(m)} \qquad (20)$$

where $\mathbf{N}^{(m)}$ is called the frequency dispersion matrix, whose elements are,

$$N_{kl}^{(m)} = \int_0^\infty D(r)W(r)f_k^{(m)}(r)f_l^{(m)}(r)rdr \qquad (21)$$

Note that $\mathbf{M}^{(m)}$ is proportional to the beam intensity, therefore, at very low intensity, the system is dominated solely by $\mathbf{N}^{(m)}$). This matrix is real and symmetric, which decomposes the original system into several lossless resonators with slightly difference inherent oscillation frequencies, and introduces no instability mechanism. When the beam intensity increases, the matrix $\mathbf{M}^{(m)}$ becomes dominant, and the system shows the stability or instability. The transition of the two different status shows the property of Landau damping. The equation (20) can be used for study of Landau damping in the process of antidamping, damping, and the combined situation, with a reasonable computational load [7]. To include mode coupling, $\omega_s(r)$ is substituted into the equation (15). The result is shown to be the same as (16), except the following matrix is used as the system matrix,

$$\left[\begin{bmatrix} m\mathbf{N}^{(m)} & 0 \\ 0 & m'\mathbf{N}^{(m')} \end{bmatrix} + \begin{bmatrix} \mathbf{M}^{(m,m)} & \mathbf{M}^{(m',m)} \\ \mathbf{M}^{(m,m')} & \mathbf{M}^{(m',m')} \end{bmatrix}\right] \qquad (22)$$

4.2 Dispersion Relation Type Solution

The equation (20) can also be solved by the dispersion relation [8]. Substituting $\omega_s(r)$ into (1) and moving $\omega - m\omega_s(r)$ to the right side, we get,

$$\boldsymbol{\alpha}^{(m)} = \mathbf{A}^{(m)}\mathbf{M}^{(m)}\boldsymbol{\alpha}^{(m)} \qquad (23)$$

where the elements of the matrix $\mathrm{A}^{(m)}$ are,

$$A_{kl}^{(m)} = \int_0^\infty \frac{W(r)f_k^{(m)}(r)f_l^{(m)}(r)rdr}{\omega - m\omega_s(r)} \qquad (24)$$

This matrix shows the dispersion relation. By carefully handling the singularity problem, the solution for any mode of each azimuthal mode can be obtained. To reach a comprehensive solution including all necessary modes seems to be difficult.

V. DISCUSSION

For all previously treated problems, eigenvalue type solutions exist, which provides convenience in the calculation, and also physical insight into the problem. The synchrotron frequency spread alone doesn't generate coherent motion, when the beam intensity increases, however, the feedback force becomes effective and an eigenvalue equation can be used to approximate the system by several subsystems. The convergence is guaranteed, and for most impedances the required expansion dimension is not large.

If the effect of the stationary beam distribution cannot be neglected, the formalism has to be revised. One approach [9] is to use the meshes in the r direction, which itself is an orthogonal basis. This method is straightforward but the computational load is drastically increased. It is of interest to search for more efficient methods.

Various formalisms and the related solutions of the perturbation have been briefly reviewed. All formalisms have applications under certain conditions, and also all are subjected to limitations. These conditions and limitations need to be clearly specified.

VI. REFERENCES

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