Entropy and Emittance Growth*

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ABSTRACT

In this paper we examine the connection between emittance growth and entropy growth in linear accelerators. We divide emittance growth in to two classes: reversible and irreversible depending on the corresponding entropy change. We propose the general hypothesis that if $\Delta\varepsilon > 0$ and $\Delta S = 0$, then the emittance growth may be reversible. We also propose that if $\Delta\varepsilon > 0$ and $\Delta S > 0$ then the emittance growth is irreversible. We outline how the concept may be applied to particular cases of relevance e.g. emittance growth and recovery in electron photoinjectors, and wakefield induced emittance growth, where correlations are introduced in the transverse phase space.*

INTRODUCTION

The general connection between emittance (ε) and entropy (S) was made over two decades ago by Lawson, Lapostolle and Gluckstern [1]. The connection between emittance growth and entropy growth has been mentioned briefly by some authors [2,3].

In classical thermodynamics entropy is defined in two ways [4,5]. Entropy is considered as a macroscopic quantity in equilibrium thermodynamics, or as a microscopic quantity of a statistical ensemble. In macroscopic thermodynamics, entropy, like temperature, cannot be defined for a system that is not in thermal equilibrium. This poses a difficulty when dealing with non thermalized or non equilibrium beam distributions such as those generated in photoinjectors [6].

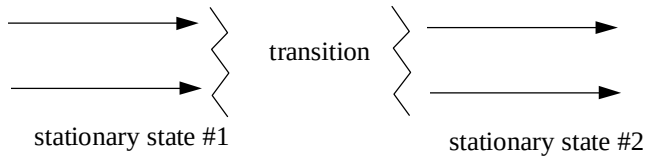

Fig. 1 Macroscopic states of a beam system

We can define both thermodynamic and statistical entropy for equilibrium state #1 or #2 in Fig.1, and consequently the difference in entropy between the two states. In the equilibrium the two definitions lead to identical results. In many cases, the transition i.e. non-equilibrium region, may extend from the cathode to the beam dump [7,8]

DETERMINING THE ENTROPY

In the transition region we must rely on the statistical definition. In the most general sense the statistical entropy of a system can be written, as:[4]

$$S = -k\sum_i f_i \ln(f_i) \quad (1)$$

where k is Boltzmann's constant, i denotes a microcanonical state of the system, and f_i is the statistical probability of that state and $\sum_i f_i = 1$.

In the case of a beam microbunch with a very large number of particles (N) we can write the sum as in integral by noting that a state corresponds to a six dimensional volume element $A_6 = \delta^6 x$, and the probability is equivalent to the product of A_6 times the distribution function $\rho_6(x,p_x,y,p_y,z,p_z)$. So that:

$$S_6 = -kN\int \rho_6(x)\ln[A_6\rho_6(x)]d^6x \quad (2)$$

where $\int \rho_6(x)d^6x = 1$. To simplify matters for this short paper, and without loss of generality, let us assume a monochromatic beam and consider a 3-D trace space (x,x',ζ) where $x' = dx/dz$ and ζ is the axial coordinate relative to the center of the bunch, and $A_3 = \delta x \delta x' \delta\zeta$. So that the 3-D entropy is given by

$$S_3 = -kN\int \rho_3(x)\ln[A_3\rho_3(x)]d^3x \quad (3)$$

We will neglect the numerical subscripts from now on. To deal with beams that are changing energy we can define a normalized entropy S_n as:

$$S_n = -kN\int \rho(x,x',\zeta)\ln[\frac{A\rho(x,x',\zeta)}{\beta\gamma}]dxdx'd\zeta \quad (4)$$

while keeping A constant, independent of energy.

How do we choose the size of the volume element A? Let us neglect quantum mechanical limits. We could choose

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A based on the limits of our ability to make observations on the beam i.e. on the resolution of our instrumentation or on the limits of physical phenomena of interest.

Consider the division of the bunch in to microstates or sub-bunch slices as in Fig. 2.

We divide the bunch into a number of sub-bunches or slices i of axial length $\delta\zeta$ where $\delta\zeta \ll \sigma_b$ the bunch length and ζ_i is the axial coordinate of the i^{th} sub-bunch relative to the centroid of the bunch. Consider each sub-bunch to contain a large number of particles, N_i. The ensemble entropy of the bunch is the sum of the entropies of the sub-bunches i.e. $S = \sum_i S_i$ where S_i is the entropy of the i^{th} sub-bunch. So in our reduced dimensional space S_i is a 2-D entropy and S is the 3-D entropy.

In general the normalized sub-bunch entropy can be written as:

$$S_{i,n} = kN_i \ln\left(\frac{D_\rho \pi \tilde{\varepsilon}_{i,n} \delta\zeta}{A}\right) \quad (5)$$

where $\tilde{\varepsilon}_{i,n}$ is the normalized rms emittance of a sub-bunch, and D_ρ is a unitless parameter that depends on the shape of the trace space particle distribution. For a K-V distribution $D_{KV} = 4$; for a Gaussian distribution $D_G \approx 5.4$; for a water bag distribution $D_W = 5$. The entropy depends not only on the rms emittance but also on the distribution function.

An entropy change can occur from a change in the emittance, the distribution function or the number of particles in the sub-bunch. Note that changes in N_i correspond to axial migration of particles.

CORRELATED BEAMS

Consider in the case of a beam generated in a an rf photoinjector, in which sub-bunch correlations are introduced by the non uniform axial distribution of the bunch [9]. On scales shorter than $\delta\zeta$ we assume that randomization of the phase space occurs. On scales longer than $\delta\zeta$ we assume that there may be correlations between the phase space and the location in of the slice in the bunch.

The phase space of the i^{th} sub-bunch may be characterized by the emittance $\tilde{\varepsilon}_i$, and the Twiss parameters $\hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i$ and $\hat{\alpha}_i^2 + 1 = \hat{\beta}_i \hat{\gamma}_i$. Now envision process where the phase space distribution of beam evolves while each sub-bunch continues to contain the same particles and where the ε_i remain constant but where the Twiss parameters change in a ordered fashion i.e. the α_i, β_i, and γ_i maintain some well-defined, well behaved functional relationships such that:

$$\hat{\alpha}_i = \hat{\alpha}(\zeta, z),\ \hat{\beta}_i = \hat{\beta}(\zeta, z),\ \text{and } \hat{\gamma}_i = \hat{\gamma}(\zeta, z) \text{ and } \tilde{\varepsilon}_i = \tilde{\varepsilon}(\zeta, z)$$

where z is the coordinate of the bunch centroid along the beamline in the laboratory frame.

If the phase space is correlated in ζ, then the forces that determine the phase space evolution must also be correlated in ζ. Other examples of such processes might include single bunch transverse-wakefield induced head to tail kicks, and phase dependent longitudinal and transverse rf effects.

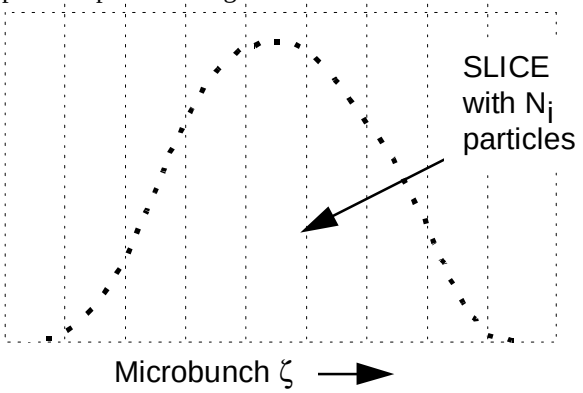

Fig. 2 Division of the bunch in to slices.

Consider a longitudinal density distribution function $\rho(\zeta)$ independent of z, such that $\int \rho(\zeta) d\zeta = 1$. If we define the normalized rms emittance of the bunch in the conventional way as:

$$\tilde{\varepsilon}_n(z) = \beta\gamma\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right]^{0.5}$$

then:

$$\tilde{\varepsilon}_n(z) = \left[\int \tilde{\varepsilon}_n(\zeta, z)\hat{\beta}(\zeta, z)\rho(\zeta)d\zeta \int \tilde{\varepsilon}_n(\zeta, z)\hat{\gamma}(\zeta, z)\rho(\zeta)d\zeta - \left(\int \tilde{\varepsilon}_n(\zeta, z)\hat{\alpha}(\zeta, z)\rho(\zeta)d\zeta\right)^2\right]^{0.5} \quad (6a)$$

At z = 0, i.e. at the cathode, the emittance is the same for all slices, so we can write $\tilde{\varepsilon}_n(\zeta, 0) = \tilde{\varepsilon}_{s,n}$, where $\tilde{\varepsilon}_{s,n}$ is the slice emittance at the cathode. Consider a situation where the emittance of each slice is independent of z. This will be true for linear transverse forces acting on each slice. Then:

$$\tilde{\varepsilon}_n(z) = \tilde{\varepsilon}_{s,n}\left[\int \hat{\beta}(\zeta, z)\rho(\zeta)d\zeta \int \hat{\gamma}(\zeta, z)\rho(\zeta)d\zeta - \left(\int \hat{\alpha}(\zeta, z)\rho(\zeta)d\zeta\right)^2\right]^{0.5} = \tilde{\varepsilon}_{s,n} C(z) \quad (6b)$$

where we call C(z) the emittance correlation function and C(0) = 1 i.e. at the cathode the bunch phase-space is uncorrelated. Because of the properties of the Twiss parameters $C(z) \geq 1$ always. Therefore $\tilde{\varepsilon}_n(z) \geq \tilde{\varepsilon}_{s,n}$ always. The value of C(z) is calculable form the beam dynamics. We can write the normalized entropy in this case as :

$$S_n = \sum_i S_{i,n} = \sum_i kN_i \ln\left(\frac{D_\rho \pi \tilde{\varepsilon}_{s,n} \delta\zeta}{A}\right) = kN \ln\left(\frac{D_\rho \pi \tilde{\varepsilon}_{s,n} \delta\zeta}{A}\right) \quad (7)$$

the latter equality when the $\tilde{\varepsilon}_{s,n}$ are equal. Note that the entropy of the bunch, unlike the emittance, does not depend

explicitly on the precise details of the of the orientation of the phase space of each slice. Therefore it is possible for $\tilde{\varepsilon}_n(z)$ to change i.e. increase or decrease without any change in S_n.

REVERSIBLE AND IRREVERSIBLE PROCESSES

In order for S_n to increase the slice emittances $\tilde{\varepsilon}_{s,n}$ must increase. If the $\tilde{\varepsilon}_{s,n}$ increase then from eqn. (6), $\tilde{\varepsilon}(z)$ will increase irreversibly and $\delta S_n > 0$. Such increase could be the result of non linear space-charge or other forces acting on the beam. Conversely if the $\tilde{\varepsilon}_{s,n}$ do not increase then any increase or decrease in $\tilde{\varepsilon}(z)$ is a result of changes in C(z) and consequently the process is in principle reversible, and also $\delta S_n = 0$.

In the latter case emittance growth becomes recoverable. This is the case in photoinjectors where solenoidal emittance compensation is used [9]. In this case the combination of self forces and the focusing force combine to introduce correlations into the bunch. It is important to note that in the photoinjector case there is no simple rotation in 6-D phase space that will remove the emittance growth. The correlated emittance growth may be removed by appropriate focusing of the bunch. We have developed an analytic description of the photoinjector emittance compensation process in which C(z) can be calculated [10]. When we have uniform transverse space-charge density in each slice and a well-behaved longitudinal distribution, we see that C(z) will grow and can be brought back down to unity when appropriate focusing forces are applied.

What happens when the correlations implied in eqn. (6) are removed, i.e. C(z) = 1 and $\tilde{\varepsilon}_n(z) = \tilde{\varepsilon}_{s,n}$? Then any further changes is $\tilde{\varepsilon}_n(z)$ result from changes in $\tilde{\varepsilon}_{s,n}$ and result in changes in S_i and S and are therefore irreversible. The correlations can be lost by axial particle mixing, driven by thermal or space-charge effects. If C(z) > 1 and no attempt is made to focus the beam then the correlations will be lost over some longitudinal diffusion distance z_D. When this happens the slice emittances grow to become equal to the bunch emittance and hence the correlation is lost, the entropy grows and the emittance growth becomes irreversible.

Another example is a bunch that experiences a transverse dipole kick such that the magnitude of the displacement is related to the position (ζ) within the bunch. The bunch emittance is increased. The sub-bunch emittances are unchanged. If an equal and opposite kick is applied then the bunch emittance is recovered. If before the correlate emittance growth is removed there are multiple uncorrelated kicks, of if there is particle diffusion, the emittance growth becomes irreversible.

Similar statements can be made about phase dependent rf effects.

In practice we can consider emittance growth to have two components:

1. where $\delta S = 0$
2. where $\delta S > 0$

so that $\delta\tilde{\varepsilon}(z)^2 = \delta\tilde{\varepsilon}(z)_R^{\ 2} + \delta\tilde{\varepsilon}(z)_I^{\ 2}$ where the subscripts R and I stand for reversible and irreversible respectively. In the example of eqn. (6b) we have:

$$\frac{\delta\tilde{\varepsilon}_n(z)^2}{\tilde{\varepsilon}\ (z)^2} = \frac{\delta C(z)^2}{C(z)^2} + \frac{\delta\tilde{\varepsilon}_{s,n}^{\ 2}}{\tilde{\varepsilon}_{s,n}^{\ 2}}$$

There is no guarantee that reversible emittance growth can in fact be reversed. The degree of achievable reversibility depends on out ability to apply correlated corrective forces to the bunch.

There are still many unanswered questions about the entropy emittance growth question. In future work we will show how entropy growth can lag behind emittance growth and establish quantitative criteria for the distance scales over which entropy growth occurs. A connection will be made to free-energy concepts.[3,11-13]. We will also generalize our concept to 6-D phase space.

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