The Effect of Coupling on Luminosity*

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Abstract

In a storage ring the existence of skew quadrupoles, solenoids, and other coupling elements breaks the independence of the horizontal and vertical motions. With the flat beams used in electron/positron colliding beam storage rings this coupling results in an increase in the vertical beam size with an attendant loss in luminosity. By defining a 'badness' parameter B_c the luminosity loss can be directly related to measurements of the coupling.

Introduction

The coupling of horizontal and vertical motions in colliding beam storage rings results in an unwanted increase in vertical beam size and hence in a loss of luminosity. It is useful in dealing with coupling to be able to relate how severe the luminosity degradation is for a given amount of coupling. To this end it is useful to define a 'badness' parameter B_c:

$$B_c \equiv \frac{\mathcal{L}(\mathrm{BBI}) - \mathcal{L}(\mathrm{BBI{+}Coup})}{\mathcal{L}(\mathrm{BBI})} \,, \tag{1}$$

where $\mathcal{L}(\mathrm{BBI{+}Coup})$ is the luminosity obtained with coupling present, and $\mathcal{L}(\mathrm{BBI})$ is the luminosity without coupling and only the beam–beam interaction to determine the beam size (and hence the luminosity). With this definition for B_c the condition needed so that the coupling is negligible is simply

$$B_c \ll 1 \,. \tag{2}$$

The usefulness of B_c comes when we can relate it directly to the coupling. This is the problem to be addressed in the rest of the paper.

Assuming equal beam sizes with $\sigma_Y \gg \sigma_X$ one finds[1]

$$\mathcal{L} = \frac{f\,N^2}{4\pi\sigma_X\sigma_Y}\left(1 + \left(\frac{\sigma_X \cdot \delta\theta}{\sigma_Y}\right)^2\right)^{-1/2} \,, \tag{3}$$

where N is the number of particles in each beam, σ_X and σ_Y are the beam sigmas along the principal axes, and $2\delta\theta \equiv (\theta_+ - \theta_-)$ is the angle between the beams due to the coupling. This differential rotation is not present if the opposing beams follow the same trajectory since, in this case, there is time reversal symmetry. However, with a pretzeled orbit, or with a two ring machine, the coupling each beam sees is different and the symmetry is lost. Using Eq. (3) in Eq. (1) and using the fact that, for weak coupling, σ_X is independent of the coupling gives

$$B_c \approx \frac{\sigma_Y^*(\mathrm{BBI{+}Coup}) - \sigma_Y^*(\mathrm{BBI})}{\sigma_Y^*(\mathrm{BBI{+}Coup})} + \frac{1}{2}\left(\frac{\sigma_X^* \cdot \delta\theta^*}{\sigma_Y^*(\mathrm{BBI{+}Coup})}\right)^2 \,, \tag{4}$$

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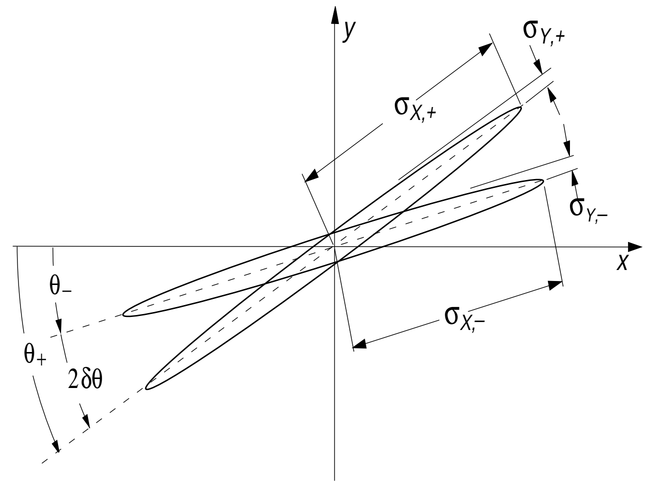

Figure 1: 1σ beam envelopes

where '$*$' indicates the quantity must be evaluated at the IP. There are two components to B_c: The first term on the RHS of Eq. (4) is due to the vertical blow-up of the beams and the second term is due to the decrease in overlap when the beams are rotated with respect to one another.

Vertical Beam Blowup

Consider first the vertical blow-up term in Eq. (4). The problem with this term is that it is not an easy matter to compute $\sigma_Y(\mathrm{BBI{+}Coup}) - \sigma_Y(\mathrm{BBI})$. The reason for this is that the beam blowup due to coupling is essentially a linear phenomena while the beam–beam induced blowup is highly nonlinear in nature. It is a nontrivial matter to determine how the beam–beam interaction couples with the coupling to affect the beam height. One option is to simply assume that the beam–beam interaction and the coupling can be taken as independent processes so that the the beam height scales in quadrature:

$$\sigma_Y^2(\mathrm{BBI{+}Coup}) = \sigma_Y^2(\mathrm{BBI}) + \sigma_Y^2(\mathrm{Coup}) \,, \tag{5}$$

where $\sigma_Y(\mathrm{Coup})$ is the vertical beam height with coupling but without the beam–beam interaction. The problem is now simpler since $\sigma_Y(\mathrm{BBI{+}Coup})$ can be approximated using the design or observed beam-beam tune shift parameter and $\sigma_Y(\mathrm{Coup})$ can be obtained from coupling data. In order to test

Eq. (5) computer simulations were performed using the weak–strong model developed by Krishnagopal and Siemann[2] modified to include coupling. The results of the simulations show more of a linear rather than a quadratic dependence. This is reasonable since the coupling changes the strength of some of the resonances driven by the beam–beam interaction. A more conservative formula would then be to take

$$\sigma_Y(\mathrm{BBI{+}Coup}) = \sigma_Y(\mathrm{BBI}) + \sigma_Y(\mathrm{Coup})\,. \qquad (6)$$

In the spirit that B_c is to be used as a first check on whether the coupling is significantly degrading the luminosity, Eq. (6) will be used. Putting Eq. (6) in Eq. (4) gives

$$B_c \approx \frac{\sigma_Y^*(\mathrm{Coup})}{\sigma_Y^*(\mathrm{BBI{+}Coup})} + \frac{1}{2}\left(\frac{\sigma_X^* \cdot \delta\theta^*}{\sigma_Y^*(\mathrm{BBI{+}Coup})}\right)^2 . \qquad (7)$$

The computation of $\sigma_Y(\mathrm{Coup})$ is relatively straightforward. The normal mode coordinate transformation for the 4x4 coupled one–turn transfer matrix $\mathbf{T}$ is written as[3, 5]

$$\begin{aligned}\mathbf{T} &= \mathbf{V}\cdot\mathbf{U}\cdot\mathbf{V}^{-1} \\ &= \begin{pmatrix} \mathbf{I}\gamma & \mathbf{C} \\ -\mathbf{C}^\dagger & \mathbf{I}\gamma \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{I}\gamma & -\mathbf{C} \\ \mathbf{C}^\dagger & \mathbf{I}\gamma \end{pmatrix}\,,\end{aligned} \qquad (8)$$

where $\mathbf{I}$ is the identity matrix, '$\dagger$' denotes the symplectic conjugate, and γ is given by $\gamma^2 + ||\mathbf{C}|| = 1$. Eigenmode a is the nearly horizontal mode and b is the nearly vertical mode. To remove the beta dependence $\mathbf{a}$ can be transformed to $\overline{\mathbf{a}}$ via

$$\overline{\mathbf{a}} = \mathbf{G}\,\mathbf{a}\,, \qquad (9)$$

where

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_b \end{pmatrix},\ \mathbf{G}_a \equiv \begin{pmatrix} \frac{1}{\sqrt{\beta_a}} & 0 \\ \frac{\alpha_a}{\sqrt{\beta_a}} & \sqrt{\beta_a} \end{pmatrix}\,, \qquad (10)$$

and similarly for $\mathbf{G}_b$ where β_a is the beta for eigenmode a. $\mathbf{T}$ is now written in terms of the normalized normal modes as

$$\mathbf{T} = \mathbf{G}^{-1}\,\overline{\mathbf{V}}\,\overline{\mathbf{U}}\,\overline{\mathbf{V}}^{-1}\,\mathbf{G}\,, \qquad (11)$$

where

$$\begin{aligned}\overline{\mathbf{V}} &= \mathbf{G}\,\mathbf{V}\,\mathbf{G}^{-1} \\ &= \begin{pmatrix} \mathbf{I}\,\gamma & \mathbf{G}_a\,\mathbf{C}\,\mathbf{G}_b^{-1} \\ -\mathbf{G}_b\,\mathbf{C}^\dagger\,\mathbf{G}_a^{-1} & \mathbf{I}\,\gamma \end{pmatrix} \\ &\equiv \begin{pmatrix} \mathbf{I}\,\gamma & \overline{\mathbf{C}} \\ -\overline{\mathbf{C}}^\dagger & \mathbf{I}\,\gamma \end{pmatrix}\,.\end{aligned} \qquad (12)$$

Since the coupling is weak the following approximations can be made:

$$\beta_a = \beta_{X,a} = \beta_x\,, \qquad \beta_b = \beta_{Y,b} = \beta_y\,, \qquad \epsilon_a = \epsilon_x\,, \qquad (13)$$

where β_x and β_y are the horizontal and vertical betas without coupling, and $\beta_{X,a}$ and $\beta_{Y,b}$ are the betas for the a and b modes projected onto the X and Y axes respectively with X and Y lying along the principal axes of a beam (cf. figure 2).

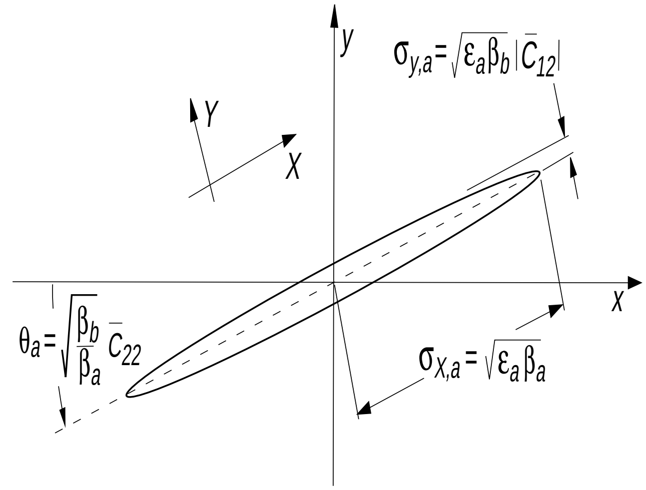

Figure 2: 1σ envelope for eigenmode a. Adapted from Bagley and Rubin figure 1.

Without the beam–beam interaction the normal mode motions are independent so the total sigma is the quadrature sum of the mode sigmas:

$$\begin{aligned}\sigma_Y^2(\mathrm{Coup}) &= \epsilon_a\,\beta_{Y,a} + \epsilon_b\,\beta_{Y,b} \\ &\equiv \sigma_{Y,a}^2 + \sigma_{Y,b}^2\,,\end{aligned} \qquad (14)$$

where ϵ_a and ϵ_b are the emittances for the normal modes

Consider first the a eigenmode. Since $\epsilon_a \gg \epsilon_b$ the motion due to the a mode dominates so, to a good approximation, $\sigma_x = \sigma_{X,a}$ and the Y–axis coincides with the minor axis of the a mode. From figure 2, which is adapted from Bagley and Rubin[3] figure 1, we have

$$\sigma_{Y,a} = \sqrt{\epsilon_a\beta_b}\,|\overline{C}_{12}|\,, \qquad (15)$$

For the b motion $\sigma_{Y,b}$ is calculated from Eq. (14):

$$\sigma_{Y,b} = \sqrt{\epsilon_b\beta_b}\,. \qquad (16)$$

Combining Eqs. (14), (15), and (16), and using Eq. (13) gives at the IP

$$\sigma_Y^*(\mathrm{Coup}) = \sqrt{\epsilon_a\beta_y^*}\,\left(\overline{C}_{12}^{*2} + \frac{\epsilon_b}{\epsilon_a}\right)^{1/2}\,. \qquad (17)$$

With knowledge of the $\overline{\mathbf{C}}$ matrix around the ring one can calculate ϵ_a/ϵ_b[4, 6] and hence $\sigma_Y^*(\mathrm{Coup})$.

How does the contribution to σ_Y^* from $\sigma_{Y,a}^*$ and $\sigma_{Y,b}^*$ compare? Both $\sigma_{Y,a}$ and $\sigma_{Y,b}$ scale linearly with $\overline{\mathbf{C}}$ in the sense that if $\overline{\mathbf{C}}$ around the ring is scaled by some factor then both $\sigma_{Y,a}$ and $\sigma_{Y,b}$ will be scaled by the that factor[1]. However, it is important to note that $\sigma_{Y,b}^*$ is dependent upon the coupling matrix around the ring as opposed to $\sigma_{Y,a}^*$ which is determined solely by the coupling matrix at the IP. Thus, it is always possible to make the a mode contribution to σ_Y^* equal to zero by using a single skew quad but the b mode contribution will always be present unless the ring is totally ('locally') decoupled.

Ignoring the tilt term for the moment, the calculation of B_c from Eqs. (7) and (17) and from knowledge of the coupling is straight forward if somewhat cumbersome. If one only wants a rough number, one can first assume that the $\sigma_{Y,a}$ contribution has been zeroed out using a skew quad. It can be shown that[1]

$$\frac{\epsilon_b}{\epsilon_a} \approx 2 \left\langle \overline{C}_{12}^2 \right\rangle_s \, , \tag{18}$$

where $\langle \cdots \rangle_s$ is an average over the ring. Using this in Eqs. (7) and (17) then gives for the vertical blowup term

$$B_c \approx \sqrt{\frac{2\epsilon_x}{\epsilon_y(\mathrm{BBI+Coup})}} \left\langle \overline{C}_{12}^2 \right\rangle_s^{1/2} \, . \tag{19}$$

Using Eq. (19) along with data on $\overline{C}_{12}$[3, 7] for the Cornell Electron/positron Storage Ring CESR shows that with a modest amount of global coupling B_c can be as high as 0.3 and with local decoupling can be decreased to as low as 0.07[1]. This is in line with the qualitative observation that local decoupling is necessary to obtain the highest luminosity[7].

$\delta\theta$ Calculation

For a given beam since $\epsilon_a \gg \epsilon_b$ the a eigenmotion dominates. Therefore, with negligible error we can take the angle of a beam, θ to correspond to θ_a — the angle for the a mode ellipse. θ_a is related to $\overline{C}_{22}$ as shown in figure 2. Using this gives

$$\delta\theta^* \equiv \frac{1}{2}(\theta_+^* - \theta_-^*) = \sqrt{\frac{\beta_y^*}{\beta_x^*}} \, \delta\overline{C}_{22}^* \, , \tag{20}$$

where

$$\delta\overline{C}_{22} \equiv \frac{1}{2}(\overline{C}_{22,+} - \overline{C}_{22,-}) \, . \tag{21}$$

Since $\delta\theta^*$ depends upon the difference in the $\overline{C}_{22}^*$, with pretzeled orbits $\delta\theta^*$ may be zeroed using a single skew sextupole.

The critical $\delta\theta^*$ is defined as the angle needed to give a badness of 0.1. From Eq. (7) this is found to be

$$\delta\theta_{crit}^* = 0.46 \, \frac{\sigma_Y^*(\mathrm{BBI+Coup})}{\sigma_X^*} \, . \tag{22}$$

Combining Eq. (22) with Eq. (20) gives

$$\begin{aligned} \delta\overline{C}_{22,crit}^* &= 0.46 \sqrt{\frac{\beta_x^*}{\beta_y^*}} \, \frac{\sigma_Y^*(\mathrm{BBI+Coup})}{\sigma_X^*} \\ &= 0.46 \sqrt{\frac{\epsilon_y(\mathrm{BBI+Coup})}{\epsilon_x}} \, . \end{aligned} \tag{23}$$

Eq. (23) shows that $\delta\overline{C}_{22,crit}^*$ is independent of β_x^* or β_y^*. This is just a reflection of the fact that the $\overline{C}$'s are properly normalized. This is an important point: From measurement of the $\delta\overline{C}_{22}$ 'wave' outside of the IP one can get a sense of whether $\delta\overline{C}_{22}^*$ is too large. Unfortunately, $\overline{C}_{22}$ is hard to measure accurately[3]. However, $\overline{C}_{12}$ is relatively easy to measure and since the $\overline{\mathbf{C}}$ matrix can be represented as the superposition of two rotating phasors[5] the magnitude of the $\delta\overline{C}_{12}$ wave should be very close to the magnitude of the $\delta\overline{C}_{22}$ wave. Furthermore, for a given $\delta\overline{C}_{22}$ at any point in the ring, it is easily shown that the percentage change in the overlap integral due to a finite $\delta\theta$ is independent of the local β_x and β_y. The conclusion is that a quick visual inspection of synchrotron light signals from the bends in the arcs will give an indication of how the beams are overlapping at the IP. One must always remember, however, that it is possible for the phases to be such that there is no tilt at one point in the arcs but unacceptable tilt at the IP (or vice versa).

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