TRAPPED MODES IN THE VACUUM CHAMBER OF AN ARBITRARY CROSS SECTION

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Abstract

A recent study [1] has shown that a small discontinuity such as an enlargement or a hole on circular waveguides can produce trapped electromagnetic modes with frequencies slightly below the waveguide cutoff. The trapped modes due to multiple discontinuities can lead to high narrow-band contributions to the beam-chamber coupling impedance, especially when the wall conductivity is high enough. To make more reliable estimates of these contributions for real machines, an analytical theory of the trapped modes is developed in this paper for a general case of the vacuum chamber with an arbitrary single-connected cross section. The resonant frequencies and coupling impedances due to trapped modes are calculated, and simple explicit expressions are given for circular and rectangular cross sections. The estimates for the LHC are presented.

I. Introduction

Previous computer studies of cavities coupled to a beam pipe indicated that the impedance of small chamber enlargements exhibits sharp narrow peaks at frequencies close to the cutoff frequencies of the waveguide, see references in [1]. For a single small discontinuity, such as an enlargement or a hole, on a smooth circular waveguide, an analytical theory has been developed [1], which shows that these peaks can be attributed to trapped modes localized near the discontinuity. A trapped mode is an eigenmode of the waveguide with a discontinuity, with the eigenfrequency slightly below the waveguide cutoff, which can exist in addition to the continuous spectrum of the smooth waveguide. The existence of trapped modes depends on a relation between the conductivity of the chamber walls and a typical size of the discontinuity, and in the limit of perfectly conducting walls the trapped modes exist even for very small perturbations.

The trapped modes in a circular waveguide with many discontinuities have also been studied [2], and it was demonstrated that the resonance impedance due to N close discontinuities in the extreme case can be as large as N^3 times that for a single discontinuity. This phenomenon is dangerous for the beam stability in large superconducting proton colliders like the LHC, where the design anticipates a thermal screen (liner) with many small pumping holes inside the beam pipe. In such structures with many small discontinuities and a high wall conductivity, the trapped modes can exist and contribute significantly to the beam-chamber coupling impedances.

In the present paper we develop an analytical description of the trapped modes for a waveguide with an arbitrary single-connected cross section. We also derive particular results for circular and rectangular waveguides from our general formulas.

II. General Analysis

Let us consider a cylindrical waveguide with a transverse cross section S, having a small hole in its perfectly conducting walls. We assume that the z axis is directed along the waveguide axis, the hole is located at the point ($\vec{b}$, $z=0$), and its typical size h satisfies $h \ll b$. The fields of a source with time dependence $\exp(-i\omega t)$ in the waveguide without hole can be expressed as a series in TM- and TE-modes. The fields of the TM$_{nm}$ mode are [3]

$$\begin{aligned} E_z^{\mp} &= k_{nm}^2 e_{nm} \exp(\pm\Gamma_{nm} z)\ ; \qquad H_z^{\mp} = 0\ ; \\ \vec{E}_t^{\mp} &= \pm\Gamma_{nm} \vec{\nabla} e_{nm} \exp(\pm\Gamma_{nm} z)\ ; \\ Z_0 \vec{H}_t^{\mp} &= ik\hat{z} \times \vec{\nabla} e_{nm} \exp(\pm\Gamma_{nm} z)\ , \end{aligned} \tag{1}$$

where $\pm$ indicates the direction of the mode propagation, k_{nm}^2, $e_{nm}(\vec{r})$ are eigenvalues and orthonormalized eigenfunctions (EFs) of the 2D boundary problem in S:

$$\left(\nabla^2 + k_{nm}^2\right) e_{nm} = 0\ ; \quad e_{nm}\big|_{\partial S} = 0\ , \tag{2}$$

and propagation factors $\Gamma_{nm} = (k_{nm}^2 - k^2)^{1/2}$ are to be replaced by $-i\beta_{nm}$ with $\beta_{nm} = (k^2 - k_{nm}^2)^{1/2}$ for $k > k_{nm}$. Here $\vec{\nabla}$ is the 2D gradient in plane S; $k = \omega/c$; $\hat{\nu}$ means an outward normal unit vector, $\hat{\tau}$ is a unit vector tangent to the boundary ∂S of the chamber cross section S, and $\{\hat{\nu}, \hat{\tau}, \hat{z}\}$ form a RHS basis. Similarly, TE$_{nm}$ fields are expressed in terms of EFs h_{nm} satisfying the boundary problem (2) with the Neumann boundary condition $\nabla_\nu h_{nm}|_{\partial S} = 0$, and corresponding eigenvalues $k_{nm}'^2$ [3].

A. Frequency Shifts

In the presence of the hole, there is a solution of the homogeneous, i.e., without external currents, Maxwell equations for this structure with the frequency Ω_s ($s \equiv \{nm\}$) slightly below the corresponding cutoff frequency $\omega_s = k_s c$, so that $\Delta\omega_s \equiv \omega_s - \Omega_s \ll \omega_s$ — the sth trapped TM-mode. At distances $|z| > b$ from the discontinuity the fields of the trapped mode have the form

$$\begin{aligned} \mathcal{E}_z &= k_s^2 e_s \exp(-\Gamma_s|z|)\ ; \qquad \mathcal{H}_z = 0\ ; \\ \vec{\mathcal{E}}_t &= \mathrm{sgn}(z)\Gamma_s \vec{\nabla} e_s \exp(-\Gamma_s|z|)\ ; \\ Z_0 \vec{\mathcal{H}}_t &= ik\hat{z} \times \vec{\nabla} e_s \exp(-\Gamma_s|z|)\ , \end{aligned} \tag{3}$$

where $k = \Omega_s/c$, and the propagation constant Γ_s satisfies the equation

$$\Gamma_s \simeq \frac{1}{4}\psi_\tau \left(\nabla_\nu e_s^h\right)^2\ . \tag{4}$$

Here ψ_τ is the transverse magnetic susceptibility of the hole, cf. [4], and superscript 'h' indicates that the field is taken at the hole. Typically, $\psi_\tau = O(h^3)$, while $\nabla_\nu e_s^h = O(1/b)$, and as a result, $\Gamma_s b \ll 1$. This means that the field of the trapped mode extends in the waveguide over the distance $1/\Gamma_s$ large compared to the waveguide transverse dimension. Conditions like Eq. (4) were obtained in [1], [2] for a circular waveguide using the Lorentz reciprocity theorem, but there are other ways to derive them. For example, they follow in a natural way from a general theory for the impedances of small discontinuities [4]. In such a derivation, the physical mechanism of this phenomenon becomes clear: a tangential magnetic field induces a magnetic moment on the hole, and the induced magnetic moment support this field if the resonance condition (4) is satisfied. Thus, the mode can exist even without an external source, see in [4]. Note that the induced electric moment P_ν is negligible for the TM-mode, since $P_\nu = O(\Gamma_s) M_\tau$, as follows from Eq. (3).

The equation (4) gives the frequency shift $\Delta\omega_s$ of the trapped sth TM-mode down from its cutoff ω_s

$$\frac{\Delta\omega_s}{\omega_s} \simeq \frac{1}{32 k_s^2} \psi_\tau^2 \left(\nabla_\nu e_s^h\right)^4 \ . \tag{5}$$

In the case of a small hole this frequency shift is very small, and for the trapped mode (3) to exist, the width of the resonance should be smaller than $\Delta\omega_s$. Contributions to the width come from energy dissipation in the waveguide wall due to its finite conductivity, and from energy radiation inside the waveguide and outside, through the hole. Radiation escaping through the hole is easy to estimate [1], and for a thick wall it is exponentially small, e.g., [5]. The damping rate due to a finite conductivity is $\gamma = P/(2W)$, where P is the time-averaged power dissipation and W is the total field energy in the trapped mode, which yields

$$\frac{\gamma_s}{\omega_s} = \frac{\delta}{4k_s^2} \oint dl \left(\nabla_\nu e_s\right)^2 \ , \tag{6}$$

where δ is the skin-depth at frequency Ω_s, and the integration is along the boundary ∂S. The evaluation of the radiation into the lower waveguide modes propagating in the chamber at given frequency Ω_s is also straightforward [6], if one makes use of the coefficients of mode excitation by effective dipoles on the hole, e.g., Eqs. (6)-(9) in Ref. [4]. It shows that corresponding damping rate $\gamma_R = O(\psi_\tau^3)$ is small compared to $\Delta\omega_s$. For instance, if there is only one TE$_p$-mode with the frequency below that for the lowest TM$_s$-mode, like in a circular waveguide (H_{11} has a lower cutoff than E_{01}),

$$\frac{\gamma_R}{\Delta\omega_s} = \frac{\psi_\tau \beta_p'}{k_p'^2} \left(\nabla_\nu h_s^h\right)^2 \ , \tag{7}$$

where $\beta_p' \simeq (k_s^2 - k_p'^2)^{1/2}$ because $k \simeq k_s$.

The frequency of the trapped TE$_p$-mode is given by the condition [4]

$$\Gamma_p' \simeq \frac{1}{4}\left[\psi_z k_p'^2 \left(h_p^h\right)^2 - \chi \left(\nabla_\tau h_p^h\right)^2\right] \ , \tag{8}$$

provided the RHS of Eq. (8) is positive. Here ψ_z and χ are the longitudinal magnetic susceptibility and the electric polarizability of the hole.

B. Impedance

The trapped mode (3) gives a resonance contribution to the longitudinal coupling impedance at $\omega \approx \Omega_s$

$$Z_s(\omega) = \frac{2i\Omega_s \gamma_s R_s}{\omega^2 - (\Omega_s - i\gamma_s)^2} \ , \tag{9}$$

where the shunt impedance R_s can be calculated as

$$R_s = \frac{\sigma\delta \left|\int dz \exp(-i\Omega_s z/c)\mathcal{E}_z(z)\right|^2}{\int_{S_w} ds |\mathcal{H}_\tau|^2} \ . \tag{10}$$

The integral in the denominator is taken over the inner wall surface, and we assume here that the power losses due to its finite conductivity dominate. Integrating in the numerator one should include all TM-modes generated by the effective magnetic moment on the hole using Eqs. (6)-(9) from [4], in spite of a large amplitude of only the trapped TM$_s$ mode. While all other amplitudes are suppressed by factor $\Gamma_s b \ll 1$, their contributions are comparable to that from TM$_s$, because this integration produces the factor $\Gamma_q b$ for any TM$_q$ mode. The integral in the denominator is dominated by TM$_s$. Performing calculations yields

$$R_s = \frac{Z_0 \tilde{e}_\nu^2 \psi_\tau^3 k_s \left(\nabla_\nu e_s^h\right)^4}{8\delta \oint dl \left(\nabla_\nu e_s\right)^2} \ , \tag{11}$$

where $\tilde{e}_\nu \equiv -\sum_s e_s(0)\nabla_\nu e_s(\vec{b})/k_s^2$ is the normalized electric field produced at the hole location by a filament charge on the chamber axis, see [7] and [4].

Results for a particular shape of the chamber cross section are obtained from the equations above by substituting the corresponding eigenfunctions.

III. Circular Chamber

For a circular cross section of radius b the eigenvalues $k_{nm} = \mu_{nm}/b$, where μ_{nm} is mth root of the Bessel function $J_n(x)$, and the normalized EFs are

$$e_{nm}(r,\varphi) = \frac{J_n(k_{nm} r)}{\sqrt{N_{nm}^E}} \left\{ \begin{array}{c} \cos n\varphi \\ \sin n\varphi \end{array} \right\} \ , \tag{12}$$

with $N_{nm}^E = \pi b^2 \epsilon_n J_{n+1}^2(\mu_{nm})/2$, where $\epsilon_0 = 2$ and $\epsilon_n = 1$ for $n \neq 0$. For TE-modes, $k_{nm}' = \mu_{nm}'/b$ with $J_n'(\mu_{nm}) = 0$, and

$$h_{nm}(r,\varphi) = \frac{J_n(k_{nm}' r)}{\sqrt{N_{nm}^H}} \left\{ \begin{array}{c} \cos n\varphi \\ \sin n\varphi \end{array} \right\} \ , \tag{13}$$

where $N_{nm}^H = \pi b^2 \epsilon_n (1 - n^2/\mu_{nm}'^2) J_n^2(\mu_{nm}')/2$. In this case $\tilde{e}_\nu = 1/(2\pi b)$, which follows from the Gauss law. Assuming the hole located at $\varphi = 0$, we get from Eq. (4)

$$\Gamma_{nm} = \frac{\psi_\tau \mu_{nm}^2}{2\pi\epsilon_n b^4} \ , \tag{14}$$

and from Eq. (11)

$$R_{nm} = \frac{Z_0 \psi_\tau^3 \mu_{nm}^3}{32\pi^4 \epsilon_n \delta b^8} \ . \tag{15}$$

For TE-modes from Eq. (8)

$$\Gamma'_{nm} = \frac{\psi_z \mu'^4_{nm}}{2\pi\epsilon_n b^4(\mu'^2_{nm} - n^2)} \ . \tag{16}$$

Note that only the modes with $\cos n\varphi$ can be trapped, while $\sin$-modes just do not "see" the hole.

The results of this section coincide with those of [1], [2], except R_s in [1], where the contribution of only the trapped mode to Eq. (10) was taken into account. Formulas for an axisymmetric enlargement with area A of the longitudinal cross section are easily obtained from Eqs. (14)-(15) with $n = 0$ by substitution $\psi_\tau \to 4\pi b A$.

IV. Rectangular Chamber

For a rectangular chamber of width a and height b the eigenvalues $k_{nm} = \pi\sqrt{n^2/a^2 + m^2/b^2}$ for $n, m = 1, 2, \ldots$, and the normalized EFs are

$$e_{nm}(x, y) = \frac{2}{\sqrt{ab}} \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{b} \ , \tag{17}$$

with $0 \le x \le a$ and $0 \le y \le b$. Let a hole be located in the side wall at $x = a, \ y = y_h$. Then Eq. (4) gives

$$\Gamma_{nm} = \frac{\psi_\tau \pi^2 n^2}{a^3 b} \sin^2\left(\frac{\pi m y_h}{b}\right) \ , \tag{18}$$

and from Eq. (11) the impedance is

$$R_{nm} = \frac{Z_0 \psi_\tau^3 \pi^3 n^2 \sqrt{n^2 b^2 + m^2 a^2}}{2\delta a^4 b^2 (n^2 b^3 + m^2 a^3)} \Sigma^2\left(\frac{a}{b}, \frac{y_h}{b}\right) \sin^4\left(\frac{\pi m y_h}{b}\right) \tag{19}$$

where

$$\Sigma(u, v) = \sum_{l=0}^{\infty} \frac{(-1)^l \sin[\pi(2l+1)v]}{\cosh[\pi(2l+1)u/2]} \tag{20}$$

is a fast converging series; see pictures in [7]. Both the frequency shift and especially the impedance decrease very fast if the hole is displaced closer to the corners of the chamber, i.e. when $y_h \to b$ or $y_h \to 0$.

V. Estimates

In a vacuum chamber with many discontinuities their mutual interaction is very important. For trapped modes in a circular pipe this interaction was studied in [2], but the results are applicable for any cross section of the chamber. A few holes in one cross section work as a single combined discontinuity. If the average distance g between adjacent cross sections with holes is shorter than $1/\Gamma_s$, the number of the cross sections with holes which work as an effective combined discontinuity is $N_{eff} = \sqrt{2/(\Gamma_s g)}$. Referring to [2] for more detail, in this case we use the following estimate for the reduced impedance of a cyclic accelerator due to the trapped modes

$$Re\,\frac{Z}{n} = \frac{4\pi}{\Gamma_s k_s g^2} R_s \ , \tag{21}$$

where Γ_s, k_s, and R_s are given by the formulas above.

For the LHC liner we consider a model having a square cross section with side $a = 36$ mm. The liner wall has thickness $t = 1$ mm and the inner copper coating. There are 666 narrow longitudinal slots with width $s = 1.5$ mm and length $s = 6$ mm per meter of the liner, with $M = 8$ slots in one cross section, which makes spacing $g = 12$ mm. The slots are located at distance $a/4$ from corners. Using $\psi_\tau = w^2 s/\pi$ for a long slot in the thick wall [8], we get for the lowest E-mode (TM_{11}) near 5.9 GHz

$$Re\,\frac{Z}{n} = 9.2\sqrt{RRR}\ \text{Ohms} \ , \tag{22}$$

where $RRR = 30 - 100$ for copper. The estimate for the model with the circular cross section of radius $b = 18$ mm was $16.5\sqrt{RRR}$ Ohms [2]. These estimates presume identical slots. A distribution of slot areas/lengths reduces $Re\,Z/n$ significantly: e.g., for the Gaussian distribution with RMS $\sigma_A/A_{ave} = 0.1$, the above result $16.5\sqrt{RRR}$ Ohms turns into 7 Ohms, independent of RRR, see [2].

VI. Conclusions

The trapped modes in waveguides with an arbitrary single-connected cross section are considered. The formulas for the frequency shift and the resonance impedance are derived in a general case, and the results for circular and rectangular chambers are given.

The transverse coupling impedance due to trapped modes is calculated in a similar way, see formulas for the case of a circular chamber in [2].

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