ENTROPY AND EMITTANCE OF PARTICLE AND PHOTON BEAMS*

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Abstract

The emittance as the available phase space area is defined as the product of the elementary cell area $\delta\Omega$ and $\exp(S)$, where S is the normalized entropy of a particle beam. The definition is based on the fact that the factor $\exp(S)$ can be interpreted as the number of the occupied cells. For particle beams, a closed expression for the emittance in terms of the phase space distribution function is obtained which is independent of $\delta\Omega$. To compute the emittance of the radiation beam, it is necessary to find the eigenvalues of the correlation operator. An explicit solution is found for the case of a partially coherent radiation beam which is a stochastic superposition of coherent Gaussian beams with a Gaussian probability distribution. Such a beam is a reasonable model for undulator radiation by beam of electrons. From the requirement that the radiation emittance reproduces the particle beam emittance in the incoherent limit, the elementary cell area $\delta\Omega$ is determined unambiguously to be λ, the radiation wavelength. The emittance in the coherent limit then becomes λ.

I. INTRODUCTION

The macrosopic state of a particle beam is specified by the distribution function in phase space. However, it is often useful to have a global characterization of the beam quality by means of a few numbers. The area of phase space occupied by the beam, called the emittance, is a good representation of the beam quality as it gives a measure of the uncertainty in the state of a particle beam. It is furthermore invariant under linear beam transport transformation. On the other hand, definition of emittance has been rather arbitrary.

Let us first recall several definitions of emittance. Throughout this paper, we will, for simplicity limit our discussion to the phase space distribution in one transverse direction. Let the probability distribution function in phase space be $f(x,\phi)$, with the normalization $\int f(x,\phi)dxd\phi = 1$. Examples of the possible emittance definitions are:

1. The geometric emittance, ϵ_F defined to be the area of the phase space region containing a fraction F of the total particles.
2. The peak emittance, defined as $\epsilon_0 = 1/f(0,0)$
3. The rms emittance
   $\epsilon_{rms} = \sqrt{< x^2 >< \phi^2 > - < x\phi >^2}$,
   where quantities within the angular brackets are the average values.

Each of the above definitions is suitable for certain phase space distributions, but not for others; ϵ_F could depend sensitively on the chosen value of the fraction F and could also be ambiguous for a complicated distribution, ϵ_0 is suitable only for a distribution with a well defined peak at the origin and no other places, and ϵ_{rms} is not suitable in for general non-Gaussian distributions. Also ϵ_F and ϵ_0 cannot be generalized to radiation beams. Although ϵ_{rms} can be generalized to radiation beam, it gives an infinity for the case of a coherent beam after an aperture.

This paper is an attempt to put the emittance concept on a firmer theoretical basis by relating it to the entropy in statistical mechanics. We define emittance as the product of the elememtary cell area $\delta\Omega$ and $\exp(S)$, where S is the normalized entropy of a particle beam. The definition is reasonable because the factor $\exp(S)$ can be interpreted as the number of the occupied cells. The approach provides a well defined, unified description of the beam qualities for particle and radiation beams. Such a unified understanding will be useful in describing the partially coherent beams from electron beams travelling through undulators in modern synchrotron radiation facilities.

Entropy as a measure of the quality of particle beam has been suggested before[1]. Here we provide a quantitative connection of the entropy to emittance for particle as well as radiation beam.

II. ENTROPY AND EMITTANCE

To compute the entropy of a particle beam, we divide the phace space area occupied by the beam into a large number M of elementary cells of an area $\delta\Omega$. Let N be the total number of prticles in the beam, n_k is the number of the particles in the kth cell, and $p_k = n_k/N$ be the probability that a particle occupy the kth cell. The number of ways in which the particles can be partitioned into different cells to produce a given phase space distribution is

$$P = \frac{N!}{n_1!n_2!...n_M!}. \tag{1}$$

The entropy of the beam is given by $\ln P$. The *normalized* entropy, S, is obtained by dividing the entropy by N:

$$S \equiv \frac{1}{N}\ln P = -\sum_{k=1}^{M} p_k \ln p_k. \tag{2}$$

In the above we are assuming that N and n_k are large so that Stirling's formula is applicable.

The entropy has a well known meaning as a measure of the disorder in statistical mechanics, or as the information capacity in information theory. We will relate the entropy to emittance by noting that the quantity $\exp S$ can be interpreted as the number of the occupied cells. To see this, we construct a uniform phase space distribution associated with the original distribution such that the entropies of the two distributions are the same. The number of elementary cells in the associated distribution, denoted by $\overline{M}$, will be the number of the occupied cells in the orginal distribution. The normalized entropy for the associated distribution is $\ln \overline{M}$ which by construction is equal to S. Thus we see, indeed, that $\exp S$ is the number of the occupied cells.

Based on these considerations, we write the emittance as follows:

$$\epsilon = \delta\Omega \exp S. \tag{3}$$

*This work was supported by the Director, Office of Energy Research, Office of Basic Energy Sciences, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

Equation(3) is the emittance definition based on statistical mechanics adopted in this paper. In the following sections, the emittance of particle and radiation beams are computed based on this formula.

III. PARTICLE BEAMS

For a beam of non-interacting particles, the phase space distribution function $f(x_e, \phi_e)$ is a non-negative, physically measurable quantity. Here, and in the rest of the paper, the subscript e is used to distinguish the particle (electron) variables from those of radiation. We consider first the case where the distribution is a smooth function so that for a sufficiently small value of $\delta\Omega$, Eq.(2) can be replaced by the following integral [1]:

$$S = -\int dx_e d\phi_e f(x_e, \phi_e) \ln\left(f(x_e, \phi_e)\delta\Omega\right). \tag{4}$$

The emittance from Eq.(3) becomes

$$\epsilon = \exp\left[-\int dx_e d\phi_e f(x_e, \phi_e) \ln f(x_e, \phi_e)\right] \tag{5}$$

Note that the emittance in this continuous limit is independent of $\delta\Omega$, as it should be. It is also important to observe that the emittance defined by Eq.(5) is conserved for any Hamiltonian beam transport system due to Liouville's theorem.

For a uniform distribution occupying a phase space area Ω, Eq.(5) gives $\epsilon = \Omega$, as expected.

Consider a Gaussian distribution

$$f(x_e, \phi_e) = \frac{1}{2\pi\sigma_{xe}\sigma_{\phi e}} \exp\Big(-\frac{x_e^2}{2\sigma_{xe}^2} - \frac{\phi_e^2}{2\sigma_{\phi e}^2}\Big), \tag{6}$$

where σ_{xe} and $\sigma_{\phi e}$ are respectively the rms widths of the particle distribution in x_e and ϕ_e. Equation(5) becomes in this case

$$\epsilon = e\, 2\pi\sigma_{xe}\sigma_{\phi e}. \tag{7}$$

The result is a factor $e \simeq 2.72$ larger than the peak emittance $1/f(0,0)$.

In many practically important cases, the distribution could have rapid variations within experimentally realizable phase space resolution $\delta\Omega$. The emittance in those cases should then be defined as

$$\epsilon = \exp\left[-\sum_k \delta\Omega \overline{f_k} \ln \overline{f_k}\right], \tag{8}$$

where $\overline{f_k}$ is the average of the distribution in the kth cell element of area $\delta\Omega$. The emittance defined by this equation will in general not be conserved even for a Hamiltonian system. For example, a smooth distribution at the beginning of a beam transport system can evolve into a highly filamented distribution due to non-linear transport elements. The emittance as defined by Eq.(8) will increase in such a case.

IV. RADIATION BEAM

For radiation beam, the phase space area and cells are abstract quantities[2],[3]. In this case, we proceed by noting that the probabilities p_k are the eigenvalues of the normalized version $\hat{\Gamma}_N$ of the correlation operator $\hat{\Gamma}$. This is similar to the case of quantum statistical mechanics where the density operator plays the role of the correlation operator. Equation(2) becomes, therefore

$$S = -Tr(\hat{\Gamma}_N \ln \hat{\Gamma}_N), \tag{9}$$

To relate the quantities appearing in the above to the field quantities, consider the frequency component $E(x)$ at a given frequency of the radiation field at a fixed longitudinal position along the optical axis. Throughout this paper, we ignore polarization and treat the field as a scalar. In general the field will be a stochasic variable. The correlation function of the radiation field is given by

$$\Gamma(x, x') = \left\langle E(x)E(x')^* \right\rangle. \tag{10}$$

The angular brakets in the above imply taking the statistical average. In terms of Dirac bra-ket notation, the correlation operator and the correlation function are related by

$$\Gamma(x, x') = \langle x|\hat{\Gamma}|x'\rangle. \tag{11}$$

The normalized correlation matrix is given by

$$\Gamma_N(x, x') = \frac{\Gamma(x, x')}{\int dx\, \Gamma(x, x)}. \tag{12}$$

V. PARTIALLY COHERENT UNDULATOR RADIATION

A. Model for Undulator Radiation

We now apply the above formalism to radiation generated by a beam of electrons from an undulator. To permit analytical calculation, the expression for the radiation field is simplified as follows[4]: The fundamental frequency component at $\omega = \omega_1$ from a single electron with a transverse coordinate x_e from the center of the undulator gap and with an angle ϕ_e with repesct to the undulator axis entering the undulator at time t_e can be approximately represented as follows [4]:

$$\begin{aligned} E(x; x_e, \phi_e) &= \left(\frac{I_0}{\sigma_{xr}\sqrt{2\pi}}\right)^{1/2} \exp\Big[-\frac{(x-x_e)^2}{4\sigma_{xr}^2} + \\ &\quad i\, k_1\phi_e(x-x_e) - i\omega_1 t_e\Big], \end{aligned} \tag{13}$$

where $k_1 = \omega_1/c$ is the reference wave number, and where σ_{xr} is a measure of the spread in the x-direction of the radiation produced by the single electron. The quantity I_0 is a normalization constant, defined by $\int |E(x)|^2\, dx = I_0$. Note that the field $E(x; x_e, \phi_e)$is related to $E(x; 0, 0)$ by translation of the phase space coordinates. We define the Fourier transform of the radiation field by

$$\tilde{E}(\phi; x_e, \phi_e) = \int \frac{dx}{\sqrt{2\pi}} E(x; x_e, \phi_e) \exp(-i\phi k_1 x), \tag{14}$$

so that

$$\begin{aligned} \tilde{E}(\phi; x_e, \phi_e) &= \left(\frac{I_0}{k_1\sigma_{\phi r}\sqrt{2\pi}}\right)^{1/2} \exp\Big[-\frac{(\phi-\phi_e)^2}{4\sigma_{\phi r}^2} - \\ &\quad ik_1 x_e(\phi-\phi_e) - i\omega_1 t_e\Big], \end{aligned} \tag{15}$$

where $\sigma_{\phi r}$ is the width of the radiation field in the variable ϕ.

The radiation widths σ_{xr} and $\sigma_{\phi r}$ satisfy

$$\sigma_{xr}\sigma_{\phi r} = \frac{1}{2k_1}, \tag{16}$$

so that the Gaussian beam of Eq.(13) is a minimum uncertainty wave packet. For an undulator of length L, we have $\sigma_{xr} = \sqrt{2\lambda_1 L}/4\pi$ and $\sigma_{\phi r} = \sqrt{\lambda_1/2L}$, where $\lambda_1 = 2\pi/k_1$, the wavelength corresponding to k_1.

The total electric field $E(x)$ is obtained by summing over contributions from different electrons $E(x; x_e, \phi_e)$.

B. Correlation and Entropy

In calculating the average in Eq.(10), terms involving product of electric fields from different electrons vanish due to random phase factors. The contributions from the same electron is averaged with the electron probability distribution $f(x_e, \phi_e)$. Thus,

$$\Gamma(x, x') = N_e \int dx_e\, d\phi_e\, f(x_e, \phi_e) E(x; x_e, \phi_e) E(x'; x_e, \phi_e)^* \tag{17}$$

where N_e is the total number of electrons. Assuming that the electron distribution is Gaussian as given by Eq.(6), the integrals can be performed, and we obtain

$$\Gamma_N(x, x') = \frac{1}{\Delta_x\sqrt{2\pi}} \exp\left[-\frac{1}{2\Delta_x^2}\Big(\frac{x+x'}{2}\Big)^2 - \frac{k_0^2\Delta_\phi^2(x-x')^2}{2}\right], \tag{18}$$

where

$$\Delta_x^2 = \sigma_{xe}^2 + \sigma_{xr}^2, \tag{19}$$

$$\Delta_\phi^2 = \sigma_{\phi e}^2 + \sigma_{\phi r}^2. \tag{20}$$

To compute the entropy we must carry out the trace in Eq.(9). For this purpose it is convenient to have the eigenvalues p_k of the operator $\hat{\Gamma}_N$, i.e., the eigenvalues which appear in the equation

$$\int dx'\, \Gamma_N(x, x')\phi_k(x') = p_k\phi_k(x), \tag{21}$$

where $\phi_k(x)$ are the eigenfunctions. The eigenvalues p_k are necessarily non-negative, since $\hat{\Gamma}$ is a non-negative definite operator. In fact, p_k can be identified as the probability to occupy the kth cell introduced in Section II.

It turns out that the integral eigenvalue equation, Eq.(21), can be solved by noting the similarity of the present problem with the quantum statistical mechanics of harmonic oscillators[5]. Without going into the derivation [6], we give the result for the normalized entropy for the partially coherent radiation beam as follows:

$$S = \frac{1}{2}\left[(\eta+1)\ln\frac{\eta+1}{2} - (\eta-1)\ln\frac{\eta-1}{2}\right], \tag{22}$$

where

$$\eta = 2k_1\Delta_x\Delta_\phi \geq 1. \tag{23}$$

C. Emittance of Partially Coherent Radiation Beam

The emittance is given by Eq.(3) with S determined from Eq.(22). In contrast to the particle beam case, however, we need to specify the value of the elementary cell area $\delta\Omega$. We will prove below the very reasonable result that $\delta\Omega = \lambda_1$, the wavelength of the radiation. Indeed, we note that the radiation emittance should approach the electron beam emittance when $\eta \gg 1$. In this incoherent limit, Eq.(23) becomes

$$S = \ln\eta + 1 + O(1/\eta^2). \tag{24}$$

Therefore the emittance in the incoherent limit becomes

$$\epsilon = \delta\Omega\,(e\eta/2 + O(1/\eta)) \simeq \delta\Omega e k_1\sigma_{xe}\sigma_{\phi e}. \tag{25}$$

This becomes identical to Eq.(7) if, and only if, $\delta\Omega = \lambda_1$, as was asserted.

In the limit of vanishing electron beam emittance, the radiation emittance becomes λ_1, the elemental phase space area; The radiation is completely coherent.

A measure of coherence of a partially coherent beam is the coherent fraction F_{coh}, defined as the ratio of the coherent emittance to the full emittance. We obtain $F_{coh} = \exp(-S)$, with S given by Eq.(22). In the past, the rms definition of emittance was often used, in which case $F_{coh}^{rms} = 1/\eta$. We have compared F_{coh} and F_{coh}^{rms} as a function of $\xi = \sigma_{xe}/\sigma_{xr} = \sigma_{\phi e}/\sigma_{\phi r}$, and find that they do not differ much from each other.

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