

# Stability of Trapped Ions in Electron Storage Rings in View of Parametric Resonance

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## Abstract

Stability of the trapped ions in electron storage ring was studied with a Hamiltonian formalism derived from the Mathieu equation, in which the nonlinear field of the electron beam and the Fourier component of the beam current were taken into account.

## I. Introduction

Stability of positive ions trapped ion electron beam in electron storage rings can be distinguished by the matrix method. This derives the critical mass of the trapped ions, above which the ions are stable or the ions are trapped in the electron beam. It has been observed that the trapped ions can be eliminated by the partial fill mode operation, in which some RF buckets are not filled with electron beam bunches [1,2,3]. The stability of the ions in this mode can also be distinguished by the matrix method. But this method is only applicable in case of linear field. The electric field produced by the electron beam is highly nonlinear because of the Gaussian distribution of the electron beam.

Meanwhile, the stability was studied analytically on a special partial fill mode, in which the electron beam was distributed sinusoidally along the circumference of the ring [2]. In this mode, the equation of motion of the ions can be expressed with the Mathieu equation, so that the stability or instability of the ion can be derived straightforwardly. But this method was also discussed assuming linear field.

The electric field is linear only in the region  $z \leq \sigma_z/2$  ( $z=x$  or  $y$ ), so that the ions might be still trapped in or near the electron beam even if the unstable condition is satisfied in the linear field region. In the present paper we have studied the effects of the nonlinear field with a Hamiltonian formalism derived from the Mathieu equation. In addition, the beam current was expanded in a Fourier series, so that any pattern of the beam bunch distribution can be included in the present analysis. Special attention is paid to the 1/3 fill mode being made in ESRF, which will be probably applied to SPring-8.

## II. Stability Analysis of Trapped Ions

### A. Hamiltonian Formalism

The electron beam current can be expanded in the Fourier series. The coefficients  $\epsilon_m$ 's of the Fourier component for the 1/3 fill mode are given by  $\epsilon_m = (6/m\pi)\sin(2m\pi/3)$ , where  $m$  denotes the  $m$ -th component of the revolution frequency of the electron beam. The equation of motion of the trapped ion for one Fourier component can be written as

$$\frac{d^2 u}{dt^2} + \omega_0^2 [1 + \epsilon_m \cos(m\omega_0 t + \theta_m)] f(u) = 0 \quad (1)$$

where  $u=x/\sigma_x$  or  $y/\sigma_y$ ,  $\omega_0$  is the oscillation frequency of the ion in the linear field,  $\omega_0$  is the revolution frequency of the electron beam,  $\theta_m$  is the initial phase and  $f(u)$  is the position dependence of the electric field. Introducing the following definitions

$$\omega_m = m\omega_0/2, \quad \omega_{i0} = \omega_m + \Delta\omega \quad \text{and} \quad \theta = \omega_m t, \quad (2)$$

and taking up to the third power of the nonlinear field, at first, we can write Eq.(1) as follows,

$$\begin{aligned} \frac{d^2 u}{d\theta^2} + u + [\delta + \epsilon_m(1+\delta) \cos(2\theta + \theta_m)] u \\ - k_3(1+\delta)[1 + \epsilon_m \cos(2\theta + \theta_m)] u^3 = 0, \end{aligned} \quad (3)$$

$$(\delta = 2\delta_m + \delta_m^2, \quad \delta_m = \Delta\omega/\omega_m)$$

Equation (1) is obtained from the following Hamiltonian,

$$\begin{aligned} H = \frac{1}{2} (p_u^2 + u^2) + \frac{1}{2} [\delta + \epsilon_m(1+\delta) \cos(2\theta + \theta_m)] u^2 \\ - \frac{1}{4} k_3 (1+\delta) [1 + \epsilon_m \cos(2\theta + \theta_m)] u^4, \end{aligned} \quad (4)$$

$$(p_u = \partial u / \partial \theta)$$

Introducing the following canonical transformation

$$u = \sqrt{2J} \cos(\theta + \Phi) \quad \text{and} \quad p_u = -\sqrt{2J} \sin(\theta + \Phi), \quad (5)$$

and taking the average of the Hamiltonian over  $\theta=0 \sim 2\pi$ , we find

$$\begin{aligned} \bar{H}_1 = \frac{1}{2} J \delta + \frac{1}{4} J \epsilon_m (1+\delta) \cos(\theta_m - 2\Phi) \\ - \frac{3}{8} J^2 k_3 (1+\delta) - \frac{1}{4} J^2 k_3 (1+\delta) \epsilon_m \cos(\theta_m - 2\Phi), \end{aligned} \quad (6)$$

where we have assumed  $J$  is constant over the above period. Because of no dependence on  $\theta$ ,  $\bar{H}_1$  is a constant of motion. Introducing again the following canonical transformation,

$$Z = \sqrt{2J} \cos(\Phi + \Phi_0) \quad \text{and} \quad P = -\sqrt{2J} \sin(\Phi + \Phi_0), \quad (7)$$

we obtain

$$\bar{H}_2 = A(Z^2 + P^2) + B(Z^2 - P^2) - C(Z^2 + P^2)^2 - D(Z^4 - P^4), \quad (8)$$

with  $A = \delta/4$ ,  $B = \epsilon_m(1+\delta)/8$

$$C = 3k_3(1+\delta)/32, \quad D = k_3 \epsilon_m(1+\delta)/16$$

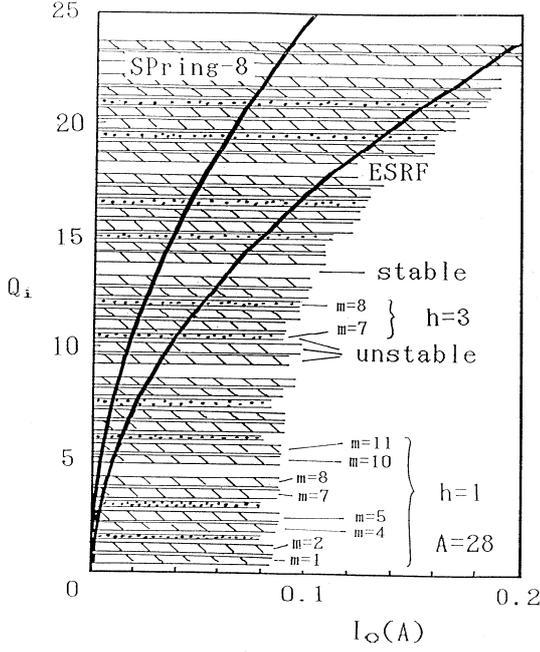


Figure 1. Stability diagram of  $\text{CO}^+$  ion in the 1/3 fill mode in ESRF and SPring-8. The ion is unstable in the shaded and dotted regions.

### B. Trajectory in the phase space and stability diagram

In case of the linear field ( $k_3=0$ ), we have  $C=D=0$ , so that the trajectory in the phase space ( $Z, P$ ) is ellipse (stable or trapped) or hyperbola (unstable), which is determined by the relation of  $A$  and  $B$ . The boundary of the stable and unstable regions is separated by the following function

$$g(\chi) = \frac{2\chi(2+\chi)}{(1+\chi)^2} = 4\chi - 6\chi^2 + 8\chi^3 \dots \quad (9)$$

with  $\chi = \delta_m$ . In case of the 1/3 fill mode the bandwidth of the unstable region can be written as

$$\delta Q_i = \epsilon_m m h / 4 = (3h/2\pi) |\sin(2\pi m/3)|, \quad (10)$$

where  $h$  is the resonant mode number of the Mathieu equation, and  $Q_i$  is defined as

$$Q_i = \frac{\omega_0}{\omega_b} = \left[ \sqrt{2\pi} \frac{r_p}{ec} \frac{I_0}{A} \frac{R^2}{\sigma_x \sigma_y} \right]^{\frac{1}{2}}, \quad (11)$$

with  $r_p$ ; the classical proton radius,  $e$ ; the charge of the electron,  $c$ ; the velocity of the light,  $I_0$ ; the average beam current,  $A$ ; the mass number of the trapped ion, and  $R$ ; the average radius of the storage ring. Figure 1 represents the stability diagram of  $\text{CO}^+$  ions in the 1/3 fill mode of ESRF and SPring-8 determined by the linear field approximation. In the figure the ordinate is  $Q_i$  and the absciss is the average beam

current. The solid curves in the figure represent the current dependence of  $Q_i$ , and the ion is unstable in the shaded and dotted regions. In this approximation the ion is unstable at any beam current.

In case of nonlinear field with  $k_3 \neq 0$ , the ion is always bounded. Examples of the trajectory in the phase space ( $Z, P$ ) are shown in Fig.2 for different value of  $\delta_m$ 's at  $\epsilon_m=0.1$ . Even at  $\epsilon_m=1$ , the trajectory is also bounded within a few  $\sigma_z$ . The results are not much different even if we add the fifth power term of the nonlinear field as shown in Fig.3. The Hamiltonian becomes very complicated if we introduce higher power terms further.

Now we consider again the stability of the trapped ion in the linear field in case of an equidistant several bunch mode operation. In case of 21 bunch mode operation in SPring-8, for instance, we have  $\epsilon_m \approx 1$  for  $m=21, 42, 63 \dots$ . Taking into account the width of the unstable region at  $\epsilon_m=1$ , we obtain the stability diagram shown in Fig.4. As shown with a solid circle in the figure, there are critical value  $Q_{ic}$  and  $I_{0c}$ , below which the ions are stable. From the analysis of the unstable region determined by the function  $g(\chi)$ , we obtain  $Q_{ic} = m^*/\sqrt{6}$ , where  $m^*$  is the minimum  $m$ , which is 21 in the present case. Meanwhile, we have from the above definition of  $Q_i$

$$Q_{ic} = \alpha \sqrt{I_{0c}/A_c} = \alpha \sqrt{I_0/A_c}, \quad (12)$$

Consequently, we find the following critical mass,

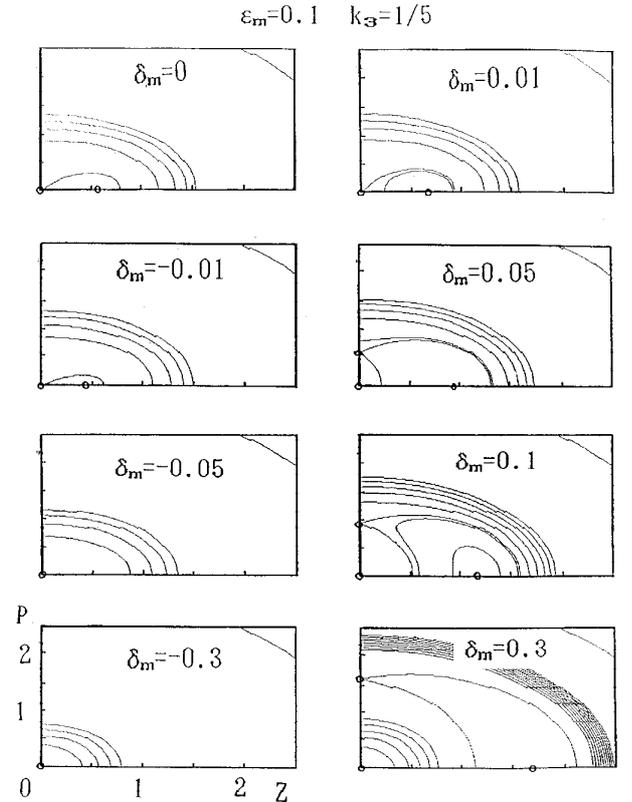


Figure 2. Ion trajectory in the phase space ( $Z, P$ ). The nonlinear field was taken into account up to the third power of the field expansion.

$$A_c \approx \frac{r_p}{\sqrt{2\pi}} \frac{N_e}{N_B} \frac{2\pi R}{N_B} \frac{1}{\sigma_x \sigma_y}, \quad (13)$$

This is very close to the critical mass derived by the matrix method.

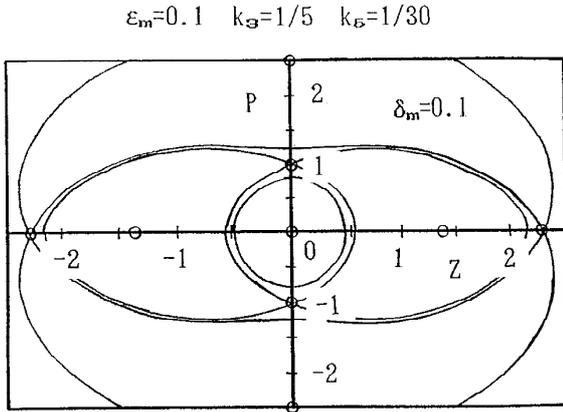


Figure 3. Ion trajectory in the phase space. The nonlinear field was taken into account up to the fifth power.

### III. Conclusion

We have discussed the stability of the trapped ions in the electron beam with the Hamiltonian formalism by taking into account the nonlinear field produced by the electron beam and the Fourier component of the beam current. It was found that the ions are bounded within a few  $\sigma$ 's of the electron beam size so long as the nonlinear field is considered up to the fifth power of the field expansion. It is required to investigate the stability by including higher power terms further. Meanwhile, in the linear field approximation, we have derived the critical mass from the present formalism, which is close to the mass derived from the matrix method. Therefore, the present formalism is more general than the matrix method, and will be useful to make a further investigation on the ion trapping.

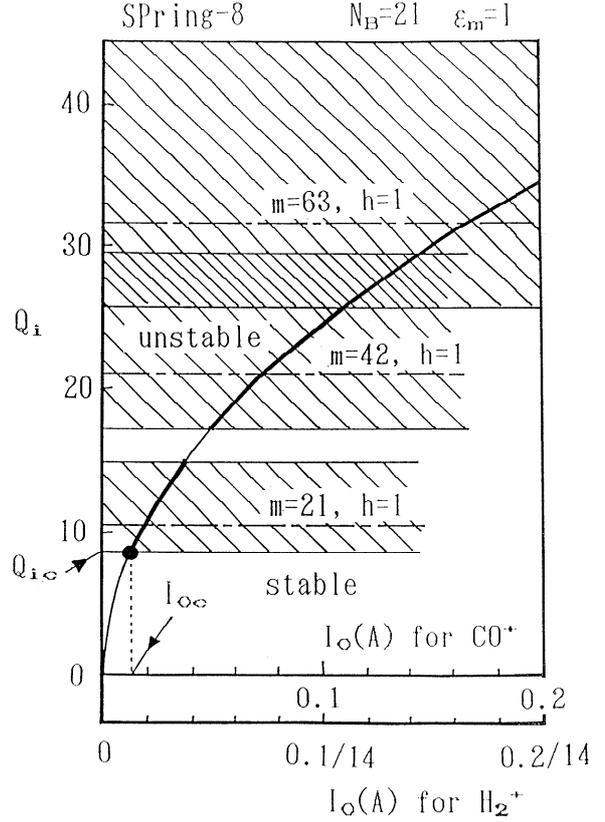


Figure 4. Stability diagram of  $\text{CO}^+$  and  $\text{H}_2^+$  ions in the equidistant 21 bunch mode of SPring-8.

### References

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