RADIATION DAMPING IN FOCUSING-DOMINATED SYSTEMS *

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Abstract

A quasi-classical method is developed to calculate the radiation damping of a relativistic particle in a straight, continuous focusing system. In one limiting case where the pitch angle of the particle θ_p is much larger than the radiation opening angle $1/\gamma$, the radiation power spectrum is similar to synchrotron radiation and the relative damping rate of the transverse action is proportional to the relative energy loss rate. In the other limiting case where $\theta_p \ll 1/\gamma$, the radiation is dipole in nature and the relative damping rate of the transverse action is energy-independent and is much faster than the relative energy rate. Quantum excitation to the transverse action is absent in this focusing channel. These results can be extended to bent systems provided that the focusing field dominates over the bending field.

I. INTRODUCTION

Radiation reaction including damping and quantum excitation has been studied extensively in synchrotrons and storage rings [1]. Recently, we demonstrated [2] that in a straight, continuous focusing channel, the radiation reaction is essentially different from that in a bending magnet. A fully quantum mechanical approach was used to investigate in detail the radiation reaction in the case $\gamma\theta_p \ll 1$ where the radiation is formed over many oscillation wavelengths. We have shown that the transverse action damps exponentially with an energy-independent damping rate, and that no quantum excitation is induced. As $\gamma\theta_p$ becomes much larger than one, the radiation is formed in a small portion of one wavelength, which can be nearly replaced by a segment of a circle. Therefore, both the radiation spectrum and the radiation damping will be similar to that from a sequence of bending magnets. In this paper, to illustrate the smooth transition between these two limiting cases, we develop a quasi-classical method to evaluate the radiation damping rate for any $\gamma\theta_p$ and obtain the expected results in both small and large $\gamma\theta_p$ limits. Then we extend these findings to focusing-dominated bent systems and consider the possibility of beam cooling based on the damping effect.

II. RADIATION

Let us consider a planar focusing system that provides a continuous parabolic potential $Kx^2/2$, where K is the focusing strength. A charged particle with energy $E = \gamma mc^2 (\gamma \gg 1)$ oscillates in the transverse x direction while moving freely in the longitudinal z direction with a constant longitudinal momentum p_z in the absence of radiation. Define the pitch angle of the particle $\theta_p = p_{x,max}/p_z$ ($p_{x,max}$ being the maximum transverse momentum) and assume that θ_p is always much less than one. The motion of the particle can be decomposed into two parts: a free longitudinal motion and a transverse harmonic oscillation; i.e., we have $E \simeq E_z + E_x$ with $E_z = \sqrt{m^2c^4 + p_z^2c^2}$ and $E_x = p_x^2c^2/2E_z + Kx^2/2$.

A quantum mechanical theory of radiation and radiation reaction for such a system was given in Ref. [2]. We only need to know that $E_x = (n + 1/2)\hbar\omega_z$, where $n = 0, 1, 2, ...$ is the transverse quantum number and $\omega_z = \sqrt{Kc^2/E_z}$ is the transverse oscillation frequency. For sufficiently large quantum number n, the transverse motion is classical and the radiation can be described by classical electrodynamics provided that the typical photon energy emitted is much smaller than the energy of the particle. Thus the energy radiated per unit solid angle per unit frequency is given by [3]

$$\frac{d^2E}{d\Omega d\omega} = \frac{e^2\omega^2}{4\pi^2 c}\left| \int_{-\infty}^{\infty} \vec{n} \times (\vec{n} \times \vec{\beta}) e^{i\omega(t' - \vec{n}\cdot\vec{r}/c)} dt' \right|^2 \quad , \qquad (1)$$

where $\vec{n}$ is the unit vector from the source to the observation point, $\vec{\beta}c$ and $\vec{r}$ are the velocity and position of the particle at the retarded time t'.

We can express Eq. (1) in the form of a double integral with respect to t_1 and t_2:

$$\frac{d^2E}{d\Omega d\omega} = \frac{e^2\omega^2}{4\pi^2 c} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 (\vec{\beta}_1 \cdot \vec{\beta}_2 - 1) e^{i(\Phi_1 - \Phi_2)}, \qquad (2)$$

where we have introduced the notation $\vec{\beta}_{1,2} = \vec{\beta}(t_{1,2})$, $\vec{r}_{1,2} = \vec{r}(t_{1,2})$ and $\Phi_{1,2} = \omega(t_{1,2} - \vec{n}\cdot\vec{r}_{1,2})$. Going over to the new variables of integration t and τ via the transformation $t_1 = t - \tau/\omega_z$ and $t_2 = t + \tau/\omega_z$, and treating the integrand in the integral with respect to t as the angular spectral distribution of the radiated power at time t, we have

$$\frac{d^3E}{dt d\Omega d\omega} = \frac{e^2\omega^2}{2\omega_z \pi^2 c} \int_{-\infty}^{\infty} d\tau (\vec{\beta}_1 \cdot \vec{\beta}_2 - 1) e^{i(\Phi_1 - \Phi_2)}. \qquad (3)$$

The averaged radiated power is obtained by integrating over $d\Omega d\omega$ and then averaging over one oscillation period (indicated by $\langle\,\rangle$). In the system we consider here, it can be shown that [4]

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{2ie^2}{\pi c}\omega_z^2\gamma^2 \int_0^{\infty} \xi d\xi \int_{-\infty}^{\infty} \frac{d\tau}{\tau} g(\tau,\xi) e^{-if(\tau,\xi)}; \qquad (4)$$

$$f(\tau,\xi) = 2\xi\tau[1 + \gamma^2\theta_p^2(1 - \sin^2\tau/\tau^2)/2],$$

$$g(\tau,\xi) = J_0(u) + \gamma^2\theta_p^2 \sin^2\tau [J_0(u) - i J_1(u)],$$

$$u = \gamma^2\theta_p^2\xi(\sin^2\tau/\tau - \sin 2\tau/2), \quad \xi = \omega/2\gamma^2\omega_z,$$

and $J_\nu(u)$ is the Bessel function of order ν ($\nu = 0, 1, 2, ...$).

Equation (4) is completely general for any $\gamma\theta_p$. The contour of integration with respect to τ must be displaced below the real axis around $\tau = 0$ to guarantee the vanishing radiation when the field is switched off [4]. The range of τ that gives a significant contribution to the integral can be defined as the

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ratio of the radiation formation length l_r to the oscillation wavelength $\lambda_z = 2\pi\sqrt{E_z/K} = 2\pi c/\omega_z$ [5]. Since l_r is of the order $\langle\rho\rangle/\gamma$ [1], where $\langle\rho\rangle$ is the averaged radius of curvature and can be approximated as $\langle\rho\rangle \sim E/(KA) \sim \lambda_z^2/A \sim \lambda_z/\theta_p$, the ratio l_r/λ_z is inversely proportional to $\gamma\theta_p$. We consider two opposite limits $\gamma\theta_p \gg 1$ and $\gamma\theta_p \ll 1$ where Eq. (4) can be greatly simplified.

In the case $\gamma\theta_p \gg 1$ or $l_r \ll \lambda_z$, using the integral representations of the Bessel functions and the method of stationary phase around $\tau = 0$, we obtain the asymptotic expression of Eq. (4) [5]

$$\left\langle\frac{dE}{dt}\right\rangle = \frac{4e^2\gamma^2\omega_z^2}{\sqrt{3}\pi c}\int_{-\pi}^{\pi}\frac{d\psi}{2\pi}\int_0^\infty \xi d\xi \int_\chi^\infty K_{5/3}(y)dy, \quad (5)$$

where $\chi = 4\sqrt{2}\xi/[3\gamma\theta_p(1-\sin\psi)^{1/2}]$ and $K_{5/3}(y)$ is the modified Bessel function of order 5/3. This expression is very similar to the frequency spectrum of synchrotron radiation [3], with χ here playing the role of ω/ω_c. Thus the equivalent critical frequency is $\omega_c \sim \gamma^3\theta_p\omega_z$, or the equivalent rotational frequency is $\omega_0 \sim c/\langle\rho\rangle \sim \theta_p\omega_z$.

In the case $\gamma\theta_p \ll 1$ or $l_r \gg \lambda_z$, expanding the integrand in Eq. (4) to leading order in $\gamma\theta_p$ and applying contour integration in the complex τ plane, we get [5]

$$\left\langle\frac{dE}{dt}\right\rangle = \frac{e^2\gamma^4\theta_p^2\omega_z^2}{c}\int_0^\infty d\xi\xi[1+2\xi(\xi-1)]\Theta(1-\xi) \quad , \quad (6)$$

where $\Theta(1-\xi)$ is the Heaviside step function. Since $\xi = \omega/2\gamma^2\omega_z$, we conclude that the radiation frequency distribution has a sharp cutoff at $\omega_d = 2\gamma^2\omega_z$, which is the characteristic of dipole radiation.

In both cases, we can carry out the integrals in Eq. (5) and (6) and find the averaged radiation power $\langle dE/dt\rangle = e^2\gamma^4\theta_p^2\omega_z^2/3c$. By using relations $\theta_p^2 \simeq 2E_x/E \simeq 2n\hbar\omega_z/E$ and $\omega_z^2 \simeq Kc^2/E$, we see that the rate of energy loss agrees with that in Ref. [2].

III. RADIATION DAMPING

The differential radiation power spectrum (i.e., Eq. (3)) can be used to define the differential number rate of photon emissions as follows: Let $R(\omega,\Omega)$ be the number of photons emitted per unit time with energies between $\hbar\omega$ and $\hbar(\omega+d\omega)$ in the directions between Ω and $\Omega+d\Omega$, i.e.,

$$R(\omega,\Omega) = \frac{1}{\hbar\omega}\frac{d^3E}{dtd\Omega d\omega} \quad , \quad (7)$$

then the average rate of change of any physical quantity, say F, is given by

$$\left\langle\frac{dF}{dt}\right\rangle = \int_0^{2\pi/\omega_z}\frac{\omega_z dt}{2\pi}\int d\omega\int d\Omega\Big|\Delta F(\omega,\Omega)\Big|R(\omega,\Omega), \quad (8)$$

where $\Delta F(\omega,\Omega)$ is the change of F after a photon with energy $\hbar\omega$ is emitted in the direction Ω. For example, the rate of energy loss $\langle dE/dt\rangle$ is obtained by replacing ΔF with $\Delta E = \hbar\omega$ and is given in the previous section.

The transverse action J_x is defined through the relation $J_x = E_x/\omega_z = (n+1/2)\hbar \simeq n\hbar$. For small change of J_x after a photon emission, we have $\Delta J_x \simeq \Delta E_x/\omega_z - \Delta\omega_z E_x/\omega_z^2$. The energy and the longitudinal momentum conservation require $\Delta E_x \simeq \hbar\omega(1-\beta\cos\theta)$[2], where θ is the angle between the photon direction and the longitudinal direction. Writing $\Delta\omega_z \simeq -\hbar\omega\omega_z/2E \simeq -\hbar\omega\omega_z\theta_p^2/4E_x$, we get

$$\Delta J_x \simeq \hbar\omega(1-\beta\cos\theta+\theta_p^2/4)\big/\omega_z \quad , \quad (9)$$

which is always positive definite. Thus the transverse action always decreases after a photon emission process and the quantum excitation is absent in such a system. This result was obtained for the transverse quantum number n based on the same kinematic argument in Ref. [2].

The damping rate of transverse action $\langle dJ_x/dt\rangle$ is obtainable by replacing ΔF in Eq. (8) with ΔJ_x in Eq. (9). With the expansion $1-\beta\cos\theta = 1/(2\gamma^2)+\theta^2/2$, $\langle dJ_x/dt\rangle$ can be written as $\langle dJ_{x1}/dt\rangle + \langle dJ_{x2}/dt\rangle$, where

$$\begin{aligned}\left\langle\frac{dJ_{x1}}{dt}\right\rangle &= \frac{1}{\omega_z}\Big(\frac{1}{2\gamma^2}+\frac{\theta_p^2}{4}\Big)\left\langle\frac{dE}{dt}\right\rangle,\\ \left\langle\frac{dJ_{x2}}{dt}\right\rangle &= \int_0^{2\pi/\omega_z}\frac{\omega_z dt}{2\pi}\int d\omega\int d\Omega\frac{1}{\omega_z}\frac{\theta^2}{2}\frac{d^3E}{dtd\Omega d\omega}. \quad (10)\end{aligned}$$

The first of the above equations is simply proportional to the rate of energy loss found in the previous section. The second one involves a more complicated angular integral. Together they account for the radiation damping for any $\gamma\theta_p$.

In the case $\gamma\theta_p \gg 1$, Eq. (10) can be simplified as [5]

$$\begin{aligned}\left\langle\frac{dJ_{x1}}{dt}\right\rangle &= \frac{1}{\omega_z}\frac{\theta_p^2}{4}\left\langle\frac{dE}{dt}\right\rangle, \quad (11)\\ \left\langle\frac{dJ_{x2}}{dt}\right\rangle &= \frac{ie^2}{2\pi c}\omega_z\gamma^2\theta_p^2\int_0^\infty \xi d\xi\int_{-\infty}^\infty \frac{d\tau}{\tau}h(\tau,\xi)e^{-if(\tau,\xi)};\\ h(\tau,\xi) &= J_0(u)+iJ_1(u)+\gamma^2\theta_p^2\sin^2\tau\Big[\frac{J_0(u)}{2}+\frac{J_2(u)}{2}\Big].\end{aligned}$$

All quantities used above are defined in Eq. (4). Similar to the calculation of the averaged radiation power, we can show [5] $\langle dJ_{x1}/dt\rangle = 2\langle dJ_{x2}/dt\rangle = e^2\gamma^4\theta_p^4\omega_z/12c$. By using the relation $J_x/E \simeq \theta_p^2/(2\omega_z)$, it is straightforward to obtain

$$\frac{1}{J_x}\left\langle\frac{dJ_x}{dt}\right\rangle = \frac{3}{4}\frac{1}{E}\left\langle\frac{dE}{dt}\right\rangle \quad . \quad (12)$$

Thus, the relative damping rate of the transverse action is proportional to the relative energy loss rate, which depends on both energy and the transverse action. This result resembles the radiation damping in a bending magnet [1], with the numerical difference due to the chromatic effect and the sinusoidal variation of the focusing field [5].

In the case $\gamma\theta_p \ll 1$, Eq. (10) becomes [5]

$$\begin{aligned}\left\langle\frac{dJ_{x1}}{dt}\right\rangle &= \frac{1}{\omega_z}\frac{1}{2\gamma^2}\left\langle\frac{dE}{dt}\right\rangle,\\ \left\langle\frac{dJ_{x2}}{dt}\right\rangle &= \frac{e^2\omega_z}{\pi c}\int_0^\infty d\xi\int_{-\infty}^\infty \frac{d\tau}{\tau^2}g(\tau,\xi)e^{-if(\tau,\xi)}. \quad (13)\end{aligned}$$

Applying contour integration again, we can show [5] $\langle dJ_{x1}/dt\rangle = \langle dJ_{x2}/dt\rangle = e^2\gamma^2\theta_p^2\omega_z/6c$. Therefore,

$$\left\langle\frac{dJ_x}{dt}\right\rangle = \frac{1}{3}\frac{e^2\gamma^2\theta_p^2\omega_z}{c} = \frac{2}{3}\frac{r_eK}{mc}J_x, \quad (14)$$

where $r_e = e^2/mc^2$ is the classical electron radius. We see that the transverse action damps exponentially with an energy-independent damping constant $\Gamma_c = 2r_eK/3mc$. An identical result for the transverse quantum number n was obtained in Ref. [2] for the "undulator regime" where $\gamma\theta_p \ll 1$. We also notice that the relative damping rate of the transverse action is much faster than the relative energy loss rate in this regime since

$$\Gamma_c = \frac{1}{J_x}\left\langle\frac{dJ_x}{dt}\right\rangle \gg \frac{1}{E}\left\langle\frac{dE}{dt}\right\rangle = \frac{\Gamma_c}{2}\gamma^2\theta_p^2 \quad . \tag{15}$$

We have shown that the radiation damping in a straight, continuous focusing channel is fundamentally different from that in a bending magnet. In the longitudinal direction the particle recoils against the emitted photon to conserve the longitudinal momentum between the two particles. However, in the transverse direction, the existence of the focusing force destroys the momentum balance and suppresses the direct recoil effect. As a result, the radiation reaction is not opposite to the photon emission direction, but always has a component pointing towards the focusing axis.

IV. FOCUSING-DOMINATED SYSTEMS

So far we have assumed that the focusing system is straight. In fact, the above discussion can be extended to bent systems under certain conditions. Consider a bent system with a constant radius ρ. A highly relativistic particle of energy E being bent by a uniform magnetic field, $B = E/ec\rho$, radiates at the rate $\langle dE/dt\rangle = 2r_ecE^4/(3m^3c^6\rho^2)$. Thus the characteristic damping (or anti-damping) rate in all three degrees of freedom due to the bending is $\Gamma_b \sim \langle dE/dt\rangle/E = 2r_ec\gamma^3/(3\rho^2)$.

In addition, the particle radiates while executing rapid betatron oscillations around the circular bent trajectory due to the focusing field. If the bending is adiabatic and the particle's pitch angle relative to the ideal orbit is small compared with $1/\gamma$, the transverse damping rate due to betatron oscillations can then be approximated by $\Gamma_c = 2r_eK/3mc$, as discussed in the previous section. Taking the ratio of these two rates, we obtain:

$$\frac{\Gamma_b}{\Gamma_c} = \frac{\bar{\lambda}_\beta{}^2}{(\rho/\gamma)^2} \quad , \tag{16}$$

where $\bar{\lambda}_\beta = \lambda_\beta/2\pi = \sqrt{E/K} = c/\omega_s$ is the reduced betatron wavelength. Equation (16) shows that if $\rho/\gamma \gg \bar{\lambda}_\beta$, the transverse damping due to local oscillations is much stronger than that from the global bending of the trajectory. Since $\rho/\gamma = E/\gamma ecB = mc/eB$, we conclude that in a system that satisfies $mc/eB \gg \sqrt{E/K}$ or $K \gg \gamma e^2B^2/m$, the radiation damping is dominated by the focusing field.

To illustrate the choice of parameters for such a system, we consider a numerical example: a focusing-dominated low energy electron ring. Let us assume that the radius of the ring is $\rho = 33$m and that $E = 0.1$GeV electrons circulate around the ring. A rather weak magnetic field $B = 0.01$T is required to confine the particles on the ideal circular trajectory. Suppose along the ideal trajectory, the electrons are continuously focused with the focusing strength $K = 30\text{GeV/m}^2$, so the reduced betatron wavelength $\bar{\lambda}_\beta$ is about 5.8cm and ρ/γ is about 17cm. From Eq. (16), we see that the transverse damping rate due to the focusing field is about nine times as fast as the characteristic damping (or anti-damping) rate from the bending field.

In a straight system, quantum excitation is absent because the transverse action must decrease after every photon emission to satisfy the kinematic constraints. In a bent system, the dispersion effect may introduce a random fluctuation of the transverse action. Nevertheless, because of the discreteness of the transverse action, there seems to exist a set of consistent conditions under which quantum excitation is prohibited even in a dispersive system [2].

V. CONCLUSION

The basic results obtained here apply to straight or bent, focusing-dominated systems. The excitation-free, asymmetric radiation damping in such systems is the direct consequence of the kinematic requirements and does not depend on the various approximations used above. There may be interesting applications of this phenomenon in beam cooling. For example, in a sufficiently low-energy, focusing-dominated electron ring, this damping effect could perhaps be utilized to obtain ultra-cool beams in transverse phase space without much energy loss. Since the system is free of radiation excitation, the actual equilibrium beam emittance will depend upon the details of the application considered.

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