Beam Distribution Function after Filamentation*

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Abstract

In this paper, we calculate the beam distribution function after filamentation (phase-mixing) of a focusing mismatch. This distribution is relevant when interpreting beam measurements and sources of emittance dilution in linear colliders. It is also important when considering methods of diluting the phase space density, which may be required for the machine protection system in future linear colliders, and it is important when studying effects of trapped ions which filament in the electron beam potential. Finally, the resulting distribution is compared with measured beam distributions from the SLAC linac.

I. INTRODUCTION

In a conservative system, which a linear accelerator or storage ring without synchrotron radiation closely approximate, the six-dimensional phase space density is conserved. Similarly, if the three degrees of freedom are uncoupled, all two-dimensional projections of the six-dimensional phase space are also conserved. A conservative emittance dilution arises when the transverse or longitudinal degrees of freedom become coupled. In this case, the 6-D emittance is preserved, but the projected emittances are increased. It can easily be shown that coupling of two planes always increases the smaller of the two projected emittances.

Because the emittance dilutions are conservative, they can be corrected, *i.e.* the the emittance can be uncoupled, provided that the dilution has not filamented (phase mixed). Filamentation arises because the beam has a spread in oscillation frequencies due to the energy spread in the beam, nonlinear fields, space charge forces, *etc.* The effect of the filamentation is to cause a phase mixing which makes it difficult to correct dilutions of the projected emittance. Once a dilution filaments, it is, for practical purposes, unrecoverable (synchrotron oscillations in a storage ring provide one obvious exception to this statement).

In this paper, we will discuss the beam distribution function arising after filamentation of a focusing mismatch. When a beam is injected into a storage ring or linac, it should be matched to the periodic or natural lattice functions. A mismatched beam will filament, with corresponding emittance growth, until it is matched to the lattice. In a storage ring, the beta function is chosen to be periodic but in a linac there is room for ambiguity since one needs to define initial values or boundary conditions. Actually, most long linacs are constructed from adiabatically varying periodic focusing cells. The natural lattice functions are simply those defined by the periodic cells.

Understanding the beam distribution function after filamentation is relevant when interpreting beam emittance measurements and locating the sources of emittance dilution. It is also important when considering methods of increasing the phase space density by deliberately mismatching the beam. Finally, it is important when studying trapped ions in an electron beam. In the next section, we will derive the distribution function for the beam action J and the projection into the x plane. Then we will present some measurements from the Stanford Linear Collider (SLC) linac, and finally, we will discuss the applications.

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II. THEORY

In a periodic linear focusing channel, a particle will perform betatron oscillations and its position and angle ($dx/ds = x'$) can be expressed in a form analogous to that of a harmonic oscillator [1]:

$$x = \sqrt{2J\beta(s)}\cos(\psi(s)+\phi) \tag{1}$$

$$x' = \sqrt{\frac{2J}{\beta(s)}}\big(\sin(\psi(s)+\phi)+\alpha(s)\cos(\psi(s)+\phi)\big) \tag{2}$$

Here, J and ϕ are the particle 'action' and 'angle' coordinates and are constants of the motion. In addition, the focusing lattice is described by the periodic lattice functions $\alpha(s)$ and $\beta(s)$ and the phase advance $\psi(s)$, where α and ψ are given by

$$\alpha \equiv -\frac{1}{2}\frac{d\beta}{ds} \qquad \psi(s) \equiv \int_0^s \frac{ds'}{\beta(s')} \; . \tag{3}$$

Finally, these equations can be inverted to solve for the action in terms of the particle coordinates

$$J = \frac{1}{2}\left(\frac{1+\alpha^2}{\beta}x^2 + 2\alpha x x' + \beta x'^2\right) . \tag{4}$$

Next, consider a particle beam that occupies some area in x-x' phase space and has a distribution function $g(x, x')$. The rms emittance of the beam is equal to

$$\epsilon \equiv \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2} \tag{5}$$

and the beam can be described with an ellipse whose orientation is specified by the second moments: $\langle x^2\rangle$, $\langle xx'\rangle$, and $\langle x'^2\rangle$, and whose area is given by $\pi\epsilon$. With complete generality, the second moments can be written in terms of the beam emittance and two parameters $\alpha^\star$ and $\beta^\star$ which we will refer to as beam parameters:

$$\langle x^2\rangle = \beta^\star\epsilon \quad \langle x'^2\rangle = \frac{1+\alpha^{\star 2}}{\beta^\star}\epsilon \quad \langle xx'\rangle = -\alpha^\star\epsilon \; . \tag{6}$$

These beam parameters $\alpha^\star$ and $\beta^\star$ describe the orientation of the beam ellipse in the (x, x') phase space and are not necessarily related to the lattice functions α and β.

The beam distribution function can be expressed in terms of the action-angle coordinates, but, in general it will depend upon both J and ϕ. Instead, we can write the position and angle of particles in terms of the beam parameters $\alpha^\star$ and $\beta^\star$ and an amplitude and phase, $J^\star$ and $\phi^\star$:

$$x = \sqrt{2J^\star\beta^\star}\cos\phi^\star \tag{7}$$

$$x' = \sqrt{2J^\star/\beta^\star}(\sin\phi^\star - \alpha^\star\cos\phi^\star) \; . \tag{8}$$

Now, assume that the beam distribution is rotationally symmetric in the normalized phase space x and $\alpha^\star x + \beta^\star x'$; this is true of bi-gaussian beams and most other distributions of interest. In this case, the distribution function will be independent of the phase $\phi^\star$ and is just a function of $J^\star$. Furthermore, the rms beam emittance ϵ is simply equal to the expectation of the amplitude $\langle J^\star \rangle$.

The action-angle coordinates can be related to the amplitude and phase as:

$$\begin{aligned} J &= J^\star\Bigg[\Bigg(\frac{\beta^\star}{\beta} + \Bigg(\alpha\sqrt{\frac{\beta^\star}{\beta}} - \alpha^\star\sqrt{\frac{\beta}{\beta^\star}}\Bigg)^2\Bigg)\cos^2\phi^\star \\ &+ 2\Bigg(\alpha - \alpha^\star\frac{\beta}{\beta^\star}\Bigg)\cos\phi^\star\sin\phi^\star + \frac{\beta}{\beta^\star}\sin^2\phi^\star\Bigg] \end{aligned} \tag{9}$$

and

$$\tan\phi = \frac{\beta}{\beta^\star}\tan\phi^\star + \Bigg(\alpha - \alpha^\star\frac{\beta}{\beta^\star}\Bigg) . \tag{10}$$

If the beam parameters are equal to the lattice functions, then the beam is 'matched' to the lattice. In this case, the action J is equal to the amplitude $J^\star$ and the angle ϕ is equal to $\phi^\star$. In addition, the beam distribution function, written in action-angle coordinates, will be independent of the angle coordinate and the rms beam emittance is equal to the expectation of the particle actions $\langle J \rangle$. If the beam filaments as it is transported through the lattice, effectively randomizing the angle coordinate ϕ, the beam emittance and distribution remain unchanged.

In contrast, if the beam is mismatched to the lattice and the beam filaments, the beam distribution function will change and the filamented rms emittance ϵ_f will increase. The emittance increase is trivially calculated from Eq. (9) and can be expressed in terms of the B_{mag} parameter [2][3]:

$$\epsilon_f = B_{\text{mag}}\epsilon , \tag{11}$$

where

$$B_{\text{mag}} \equiv \frac{1}{2}\Bigg[\frac{\beta^\star}{\beta} + \frac{\beta}{\beta^\star} + \Bigg(\alpha\sqrt{\frac{\beta^\star}{\beta}} - \alpha^\star\sqrt{\frac{\beta}{\beta^\star}}\Bigg)^2\Bigg] . \tag{12}$$

The calculation of the beam distribution function after filamentation is a little more complicated. Assuming that the angle coordinate is independent of the action after the filamentation, we can express the distribution as

$$g(J)dJ = \int_0^{2\pi}\frac{d\phi^\star}{2\pi}g^\star(J^\star)dJ^\star , \tag{13}$$

where $J^\star = J/X(\phi^\star)$ and

$$X(\phi^\star) = a\sin^2\phi^\star + 2b\sin\phi^\star\cos\phi^\star + c\cos^2\phi^\star , \tag{14}$$

with

$$a = \frac{\beta}{\beta^\star} \qquad b = \alpha - \alpha^\star\frac{\beta}{\beta^\star} \tag{15}$$

$$c = \frac{\beta^\star}{\beta} + \Bigg(\alpha\sqrt{\frac{\beta^\star}{\beta}} - \alpha^\star\sqrt{\frac{\beta}{\beta^\star}}\Bigg)^2 . \tag{16}$$

If the initial beam has a bi-gaussian distribution in x and x', then the distribution $g^\star(J^\star)$ is an exponential distribution:

$$g^\star(J^\star) = \frac{e^{-J^\star/\epsilon}}{\epsilon} , \tag{17}$$

and Eq. (13) is straight forward to evaluate. In the trivial case, where $b = 0$ and $\beta^\star \gg \beta$ or $\beta \gg \beta^\star$, the distribution for J is just a χ-squared distribution with one degree of freedom. In the general case, we can evaluate the integral by first performing a rotation to eliminate the cross term in $X(\phi^\star)$. In this case,

$$X(\phi^\star) = \lambda_1\cos^2(\phi^\star - \theta) + \lambda_2\sin^2(\phi^\star - \theta) , \tag{18}$$

where

$$\lambda_{1,2} = \frac{a+c}{2} \pm \frac{1}{2}\sqrt{(a-c)^2 + 4b^2} \tag{19}$$

$$\theta = \tan^{-1}\Bigg(\frac{c-a}{2b} - \frac{1}{2b}\sqrt{(c-a)^2 + 4b^2}\Bigg) . \tag{20}$$

By changing the variable of integration from $\phi^\star$ to $1/c^2 - 1/X$, the integral can be expressed in the form of a tabulated integral [4] and the result can be expressed in terms of the B_{mag} parameter:

$$g(J) = \frac{e^{-J B_{\text{mag}}/\epsilon}}{\epsilon}\mathrm{I}_0\Bigg(\frac{J}{\epsilon}\sqrt{B_{\text{mag}}{}^2 - 1}\Bigg) , \tag{21}$$

where I_0 is the modified Bessel function, B_{mag} is defined in Eq. (12), and ϵ is the injected beam emittance before filamentation. As expected, when $B_{\text{mag}} = 1$, the distribution is an exponential and when $B_{\text{mag}} \to \infty$, the distribution becomes a χ-squared with one degree of freedom.

Finally, we can calculate the projection into the x plane which is the beam distribution that would be measured. The projection is

$$f(x) = \int_{-\infty}^{\infty} dx' \frac{g\big(J(x,x')\big)}{2\pi} , \tag{22}$$

where $J(x,x')$ is given in Eq. (4). In the general case, the distribution function can be expressed in terms of a degenerate hypergeometric series of two variables. Unfortunately, such an expression is not any easier to evaluate than the integral Eq. (22). In the limit where $B_{\text{mag}} \to \infty$, the projection simplifies to

$$f(x) = \frac{e^{-x^2/8\beta\epsilon_f}}{\sqrt{4\pi^3}\sqrt{\beta\epsilon_f}}\mathrm{K}_0\Bigg(\frac{x^2}{8\beta\epsilon_f}\Bigg) , \tag{23}$$

where K_0 is the modified Bessel function and $\epsilon_f = B_{\text{mag}}\epsilon$. The infinite value at $x = 0$ arises because we have essentially assumed a one-dimensional injected beam. A similar expression was derived in Ref. [5] where the author was considering the distribution function for trapped ions in an electron beam.

The distribution $f(x)$ is plotted versus the rms beam size $\sqrt{\beta B_{\text{mag}}\epsilon}$ for different values of B_{mag} in Fig. 1. Notice that as the mismatch becomes larger, the relative amplitude of the central core of the beam increases while long tails contribute to the rms emittance. In the limit of large B_{mag}, the density of the core can be written:

$$\lim_{B_{\text{mag}}\to\infty} f(0) = \frac{1}{\pi^{3/2}\sqrt{B_{\text{mag}}\beta\epsilon}}\ln(\sqrt{128}B_{\text{mag}}) . \tag{24}$$

The core density decreases as $\ln(B_{\text{mag}})/\sigma_x$ rather than $1/\sigma_x$ as it would if the distribution did not change as the emittance increased.

III. MEASUREMENTS

Figure 2 shows the measured profile of a filamented beam in the SLC with large non-gaussian tails. The beam was created

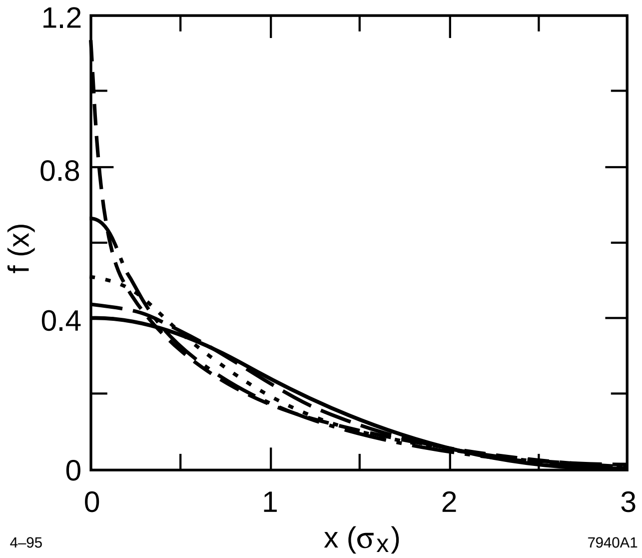

Figure. 1. $f(x)$ versus the rms beam size for $B_{\text{mag}} = 1.0$ (solid), $B_{\text{mag}} = 1.25$ (dashes), $B_{\text{mag}} = 2.0$ (dots), $B_{\text{mag}} = 5.0$ (dash-dot), and $B_{\text{mag}} = 50.0$ (dashes).

by an error in a solenoid at the low energy end of the SLAC accelerator. The beam distribution was measured after the beam had filamented; this can be determined by comparing the beam profiles measured at different betatron phases. In Fig. 2, the resulting mismatch had a $B_{\text{mag}} \approx 5$. The data was fit with a phenomenological 'super-gaussian' function [6] which shows reasonable agreement. The small asymmetry that is visible in the data is likely due to transverse wakefields.

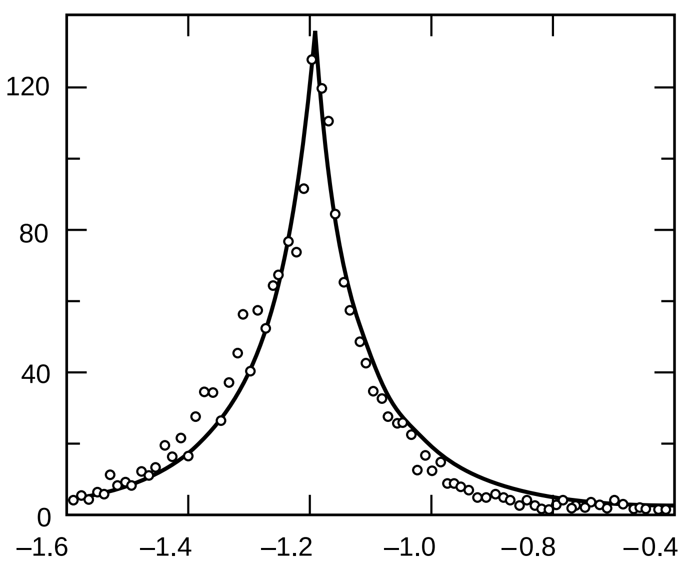

Figure. 2. Strong non-gaussian tails form a 'Christmas tree' like distribution which indicates that there is a large mismatch of the beam.

IV. DISCUSSION

In this paper, we have described the beam distribution function arising from a filamented focusing mismatch. Understanding the beam distribution can aid interpreting emittance measurements in a linear collider as well as assist in the diagnosis of the problem.

Another situation where this distribution is relevant occurs when considering machine protection schemes for future linear colliders. If mis-steered, the very small beams in future linear colliders could puncture the vacuum chamber in a single pulse. This is not a desirable feature when commissioning components. One possible method of protecting the collider is to generate a large mismatch $B_{\text{mag}} \sim 1000$ which would then filament and decrease the beam density. This approach has the advantage of not perturbing the beam centroid so that wakefield effects, steering, *etc.*, are not changed. Understanding the evolution of the core density is important in evaluating this technique.

Finally, the filamented distribution also describes the distribution of ions generated by collisional ionization and trapped in a long train of bunches [5][7]. In this case, the ions are created with a transverse density profile equal to the transverse beam profile but the ion thermal energy is typically small compared to the potential energy in beam field. Thus, the ions are mismatched relative to the focusing field of the beam. As ions continue to accumulate, the density evolves into the filamented distribution with $B_{\text{mag}} = E_{pot}/E_{th} \gg 1$ and $\beta\epsilon_f = \sigma^2/2$, where σ is the rms electron beam size.

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