FUNCTIONAL DEPENDENCE OF WAKEFUNCTIONS FOR $v < c^*$

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Abstract

In evaluating the wakefield effects of medium energy particles interacting with a cavity, the integral for the wakefunction must be carried out on the pipe radius in order to avoid an infinity due to the finite space charge effect. To obtain the wakefunctions at other radial positions, a proper extrapolation algorithm is needed. This paper presents the extrapolation method for calculating wakefunctions of $v < c$. For cases with low energies, the slippage between the particle and the fields induces longitudinal smearing of the wakefunctions, and the wakefunctions inside the beam pipe are found to be weighted averages of the wakefunctions calculated on the pipe radius. The smearing effect for calculating the wakefunctions on the axis is related to $R = \frac{\sigma_z \gamma}{a}$, with the smearing effect negligible for large R. The usual ultrarelativistic assumption is found to be reasonable for $R \geq 1.5$. For cases with $R \leq 1.5$, a weighted average must be taken to calculate the wakes inside the beam pipe.

I. INTRODUCTION

When a bunch of charged particles traverses a discontinuity in an accelerator, electromagnetic fields are excited. The particles experience an energy loss and a momentum change due to these fields. The longitudinal and transverse wakefunctions describe these effects and are defined as

$$w_l(s, \mathbf{r}) = \frac{1}{qv} \int_{-\infty}^{+\infty} dz \mathbf{v} \cdot \mathbf{E}(\mathbf{r}, z, t) \mid_{t=(z+s)/v} \quad (1)$$

$$w_\perp(s, \mathbf{r}) = \frac{1}{q} \int_{-\infty}^{+\infty} dz \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)_\perp (z, \mathbf{r}, t) \mid_{t=(z+s)/v} \quad (2)$$

In most cases, the wakefunctions are calculated under the assumption of $v = c$, which is a good approximation for high energy beams. However, there are cases of interest where the ultrarelativistic approximation is not reasonable although $v \approx c$. For example in some FEL scenarios [1], the energy of the electron beam in the accelerator is in the order of 10-10^2 MeV, and the beam bunch length is very short. In these circumstances, the effect of $v < c$ can be important.

For this medium energy range, the velocity of the particle is less than c and both the space charge and wake fields exist. The total field satisfies the inhomogeneous wave equation. The solution of the inhomogeneous equation can be separated into two parts - a special solution that satisfies the inhomogeneous equation and the general solutions that satisfy the homogeneous equation. There is a certain freedom of choosing the special solution. It is preferable, however, to chose a special solution that satisfies the $\mathbf{E}$ boundary conditions at the beam pipe radius, and the solution for such a system can be obtained analytically [2]. The special solution then represents the synchronous part (or the space charge) of the field and the general solution represents the radiated (propagating) part of the field. For cases with $v < c$, integrating the fields within the beam pipe will in general be infinite since the space charge effect is finite inside the beam pipe. In order to get meaningful results, one has to separate the effects of the synchronous field and the propagating field. The effect of the propagating fields can be separated from the synchronous field by integrating the fields at the pipe radius. The contribution of the synchronous fields at the pipe radius is zero. This is numerically advantageous in using codes like TBCI [3] and ABCI [4]. In wakefunction calculations, it is essential to integrate only the z component of the $\mathbf{E}$ field, which gives the longitudinal wakefunction. The transverse wakefunction is related to the longitudinal one through the Panofsky-Wenzel [5] theorem. Since the E_z field is nonzero only in the open gap region of the cavity, the integral at the pipe radius need only be carried out within a finite distance. The wakefunctions at other radii can be extrapolated from the one integrated at the pipe radius. The extrapolation gives the functional dependence of the wakefunction on the integral path and the smearing effects of the wakefunction due to $v < c$. (The dependence on the radial position of the source particle is explicitly calculated numerically because it is well behaved) The total effect of the fields is the summation of the extrapolated wakefunction and the space charge effect (within the length of the structure). The space charge effect can be obtained analytically, and we will not address it here. Other issues related to the numerical calculation of the wakefunction for $v < c$ include open boundary conditions at the beam pipe ends to simulate an infinitely long beam pipe and a higher-order finite-difference algorithm to reduce the frequency dependent truncation errors for short bunches. These issues are discussed in Ref. [6]. In this paper, we focus on the extrapolation of the wakefunction from the pipe radius to the inside of the beam pipe and the smearing effects due to $v < c$

II. FUNCTIONAL DEPENDENCE OF THE WAKEFUNCTION

The derivations of the functional dependence of the wakefunctions presented here assumes that the trajectories of the particles be straight lines. We study the fields that satisfy the homogeneous wave equation. For the longitudinal component of the $\mathbf{E}$ field, the equation is [7]

$$(\nabla_\perp^2 - (\zeta_z^2 - k^2)) E_z(r, \phi, z, t) = 0 \quad (3)$$

The general solution of Eq. 3 in a cylindrical coordinate system can be expressed as the follows:

$$E_z(r, \phi, z, t) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\omega d\zeta_z A(\zeta_z) G_m(k_r r) \cdot e^{-j\omega t + j\zeta_z z} e^{jm\phi} \quad (4)$$

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where

$$
\begin{aligned}
k &= \frac{\omega}{c} \\
k_r &= \sqrt{|\zeta_z^2 - k^2|} \\
G_m(k_r r) &= \begin{cases} I_m(k_r r), & \text{if} \quad \zeta_z^2 - k^2 \geq 0 \\ J_m(k_r r), & \text{if} \quad \zeta_z^2 - k^2 < 0 \end{cases}
\end{aligned}
$$

I_m and J_m are the modified Bessel function and the Bessel function of the first kind, respectively. The longitudinal wakefunction of the mth mode at $(r,\,\phi,\,s = vt - z)$ is

$$w_{l,m}(r,\phi,s) = 2\pi \int_{-\infty}^{+\infty} d\omega A(\frac{\omega}{v}) I_m(k_r r) e^{-j\frac{\omega}{v}s} e^{jm\phi} \quad (5)$$

with

$$k_r = \sqrt{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}} = \frac{\omega}{\gamma\beta c} \quad (6)$$

At the pipe radius $r = a$

$$w_{l,m}(a,\phi,s) = 2\pi \int_{-\infty}^{+\infty} d\omega A(\frac{\omega}{v}) I_m(k_r a) e^{-j\frac{\omega}{v}s} e^{jm\phi} \quad (7)$$

Fourier transforming Eq. (7), we have

$$A(\frac{\omega}{v}) = \frac{e^{jm\phi}}{(2\pi)^2 v I_m(k_r a)} \int_{-\infty}^{+\infty} w_{l,m}(a,\phi,s') e^{j\frac{\omega}{v}s} ds' \quad (8)$$

By substituting Eq. (8) into Eq. (5) we have the wakefunction at radius r

$$w_{l,m}(r,\phi,s) = \int_{-\infty}^{+\infty} ds' w_{l,m}(a,\phi,s') W_{1,m}(\gamma,a,r,s-s') \quad (9)$$

where

$$W_{1,m}(\gamma,a,r,s-s') = \frac{\gamma}{2\pi a} \int_{-\infty}^{+\infty} \frac{I_m(q\frac{r}{a})}{I_m(q)} e^{-jq\frac{(s-s')}{a/\gamma}} dq \quad (10)$$

From Panofsky-Wenzel [5] theorem, the transverse wakefunction is related to the longitudinal one as

$$w_\perp(r,\phi,s) = -\int_{-\infty}^{s} \nabla_\perp w_l(r,\phi,z') dz' \quad (11)$$

We have

$$
\begin{aligned}
w_{r,m}(r,\phi,s) &= -\frac{m}{a} \int_{-\infty}^{s} dz' \int_{-\infty}^{+\infty} ds' w_{l,m}(a,\phi,s') \cdot \\
&\quad W_{2,m}(\gamma,a,r,s-s') \quad (12) \\
w_{\phi,m}(r,\phi,s) &= -j\frac{m}{r} \int_{-\infty}^{s} dz' \int_{-\infty}^{+\infty} ds' w_{l,m}(a,\phi,s') \cdot \\
&\quad W_{1,m}(\gamma,a,r,s-s') \quad (13)
\end{aligned}
$$

with

$$
\begin{aligned}
W_{2,m}(\gamma,a,r,z'-s') &= \frac{\gamma}{2\pi a} \int_{-\infty}^{+\infty} dq \left(\frac{a}{r} \frac{I_m(q\frac{r}{a})}{I_m(q)} \right. \\
&\quad \left. + \frac{q}{m} \frac{I_{m+1}(q\frac{r}{a})}{I_m(q)} \right) e^{-jq\frac{(z'-s')}{a/\gamma}} \quad (14)
\end{aligned}
$$

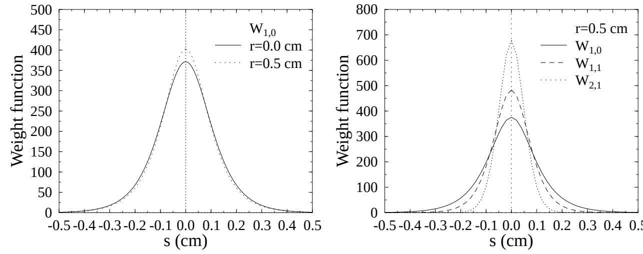

Figure 1. Weight functions for $\beta = 0.9948$, $a = 1.74$ cm.

It is clear now that the wakefunction at radius r is a weighted average of the wakefunction on the pipe radius. The weight function has finite width, which means that there is smearing effect along the s direction due to $v < c$. The profile of the weighting function is independent of the bunch length and is a function of r/a and a/γ only. The weight functions for mode $m = 0, 1$ for $\beta = 0.9948$, $a = 1.74$ cm are shown in Fig. 1. The halfwidth of the weight function for a given r/a is linear on a/γ.

At ultrarelativistic limit, the Bessel functions in the weight functions reduce to

$$\frac{I_m(q\frac{r}{\gamma})}{I_m(q\frac{a}{\gamma})} = \left(\frac{r}{a}\right)^m \quad (15)$$

The longitudinal wakefunction is independent of r for $m = 0$ and scales as r^m for other modes whereas the transverse wakefunctions scale as r^{m-1}. There is no smearing effect along the s direction.

III. THE SMEARING EFFECTS FOR $v < c$

Since the width of the short range wakefunction is roughly proportional to the bunch length, while the width of the weight function depends only on the energy and the cavity structure, the smearing effect is bunch length dependent. If the width of the weight function is much smaller than the bunch length, the smearing effect will be small, but if the width of the weight function is larger than the bunch length, the smearing effect is strong. Fig. 2 shows the wakefunctions of a 3 mm (rms) bunch with $\beta = 0.9948$ in the CEBAF 5-cell cavity at radii $r = 1.74$ cm and $r = 0$ cm. The halfwidth of the weight function for this case is 1.2 mm, which is smaller than the rms bunch length, and the smearing effect is negligible in this case. Fig. 3 shows the radial dependence of the wakefunctions of a pillbox cavity for a short bunch with bunch length of 0.5 mm (rms) and $\beta = 0.9948$ at radii $r = 1$ and $r = 0$ cm. The halfwidth of the weight function for this case is 0.66 mm, which is larger than the rms bunch length. The smearing effect is significant. The peak of the wakefunction becomes wider and lower at $r = 0$. The wakefunctions also show the slippage effects between the charge and the fields, which results in none zero wakefields ahead of the bunch.

The smearing effect of the short range wakefunction for a given energy depends not only on the energy (γ) of the beam, but also on the bunch length. The ratio

$$R = \frac{\sigma_z \gamma}{a} \quad (16)$$

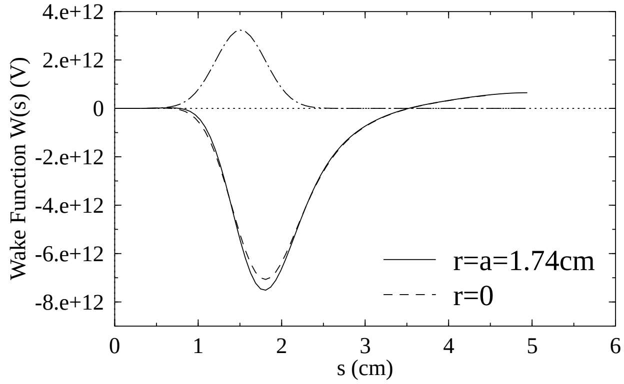

Figure 2. Wakefunctions for $\beta = 0.9948$ (5 MeV),$\sigma_z = 3$ mm. CEBAF 5-cell cavity.

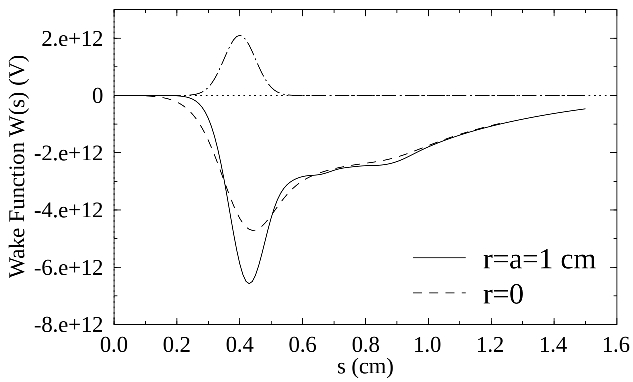

Figure 3. Wakefunctions for $\beta = 0.9948$. $\sigma_z = 0.5$ mm. Pillbox cavity.

provides a measure of the smearing effect. Large R implies weak smearing. The relative difference of the peaks of the short range wakefunctions calculated at the pipe radius and on the axis of a pillbox cavity as a function of R is shown in Fig. 4. For $R = 1.5$, the relative difference is less than 10%.

The R value of Eq.(16) can be used to determine whether the beam can be treated as ultrarelativistic in the wakefield calculation. The relative difference of the wakefunctions as a function of R may be slightly different from the one shown in Fig. 4 for different structures and bunch length. It is found from the numerical simulations that a difference of the peak of less than 10% can in general be obtained for $R > 1.5$, and the beam can be assumed ultrarelativistic. For $R < 1.5$, the smearing effect is not negligible, and wakefunctions at $r < a$ should be calculated by use of the weighted average. For example, consider the CEBAF cavity with a beam pipe of $a = 1.74$ cm. For σ_z=3 mm, the beam can be treated as being ultrarelativistic for $\gamma > 8.7$ or $E > 4.5$ MeV. For $\sigma_z = 0.5$ mm, however, the beam can be treated as being ultrarelativistic only for $\gamma > 52$ or $E > 26.5$ MeV.

IV. CONCLUSION

The effects of slippage between the beam and the fields are important in the cases of low energies and short bunches. The R value defined in Eq. (16) provides a measure of the smearing effect on the wakefields for a non-relativistic beam. The particle

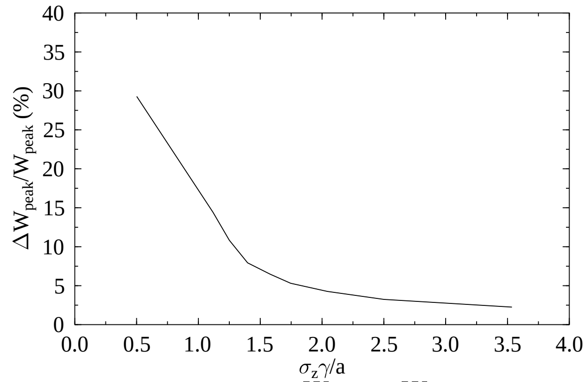

Figure 4. Relative difference $\frac{W_{pipe} - W_{axis}}{W_{pipe}}|_{peak}$ *vs.* $R = \frac{\sigma_z \gamma}{a}$. (calculated from the wakefunctions of a pillbox cavity; (r=3 cm)×(L=4 cm), pipe radius =1 cm).

can only be assumed ultrarelativistic for cases with large R.

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